Paper 150N: Fragmentation, Scale-Local, and Complement Control for High-Vorticity Navier–Stokes Amplification

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Abstract

Paper 150N continues the ordinary-channel control layer of the 150-series high-vorticity pinching program for the three-dimensional incompressible Navier–Stokes equations. Paper 150M treated the two most coherent ordinary channels: coherent aligned-patch support and transition-layer support. The present paper studies the remaining ordinary channels that may be activated when coherent structures break, become visible only after filtering, or slip across high-vorticity thresholds:

$$R_{\mathrm{frag}}, \qquad R_{\mathrm{scale}}, \qquad R_{\mathrm{low}}.$$

The purpose of Paper 150N is not to prove unconditional Navier–Stokes regularity. Its purpose is to formulate the bridge targets for fragmentation, scale-local visibility, and low-vorticity complement stretching, and to identify the analytic mechanisms by which these channels may become absorbable, lower-order, or residual pathological without exhausting the final dissipation margin.

The fragmentation channel R_{frag} represents positive stretching distributed across many separated, semi-separated, moving, reconnecting, or intermittently active components after coherent support breaks apart. The scale-local channel R_{scale} represents dangerous stretching that is not adequately visible at the full-field level but appears after filtering or scale decomposition. The low-vorticity complement channel R_{low} represents stretching that lies outside a selected high-vorticity mask, near a threshold boundary, or inside the complement of a smooth high-vorticity weight.

The target estimate is

$$R_{\mathrm{frag}}(t) + R_{\mathrm{scale}}(t) + R_{\mathrm{low}}(t) \le \delta_{\mathrm{rem,ord}} D(t) + C_{\mathrm{rem,ord}} E_\omega(t),$$

or the corresponding subinterval-stable integrated estimate

$$\int_{t_0}^{t} \big(R_{\mathrm{frag}}(s) + R_{\mathrm{scale}}(s) + R_{\mathrm{low}}(s)\big)\,\mathrm{d}s \le \delta_{\mathrm{rem,ord}} \int_{t_0}^{t} D(s)\,\mathrm{d}s + C_{\mathrm{rem,ord}} \int_{t_0}^{t} E_\omega(s)\,\mathrm{d}s + C_0.$$

These estimates are useful for the Paper 150J assembly only if the total dissipation margin remains positive:

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1.$$

The central thesis is that ordinary non-coherent exits must either pay cost, become lower-order, or become residual pathology. Fragmentation should create interface, reconnection, separation, lifetime, or alignment-loss cost. Scale-local concentration should be captured by a declared scale family with controlled overlap and finite scale budget, or else pay unresolved-gradient cost.

Complement stretching should be controlled by smooth threshold splitting, lower-order estimates, threshold-band control, or threshold-flicker accounting. If these mechanisms fail, the route must be assigned to R_{path}, where it becomes a residual pathological-channel problem for later bridge work.

Paper 150N therefore extends the bridge sequence from coherent-channel control to broken, filtered, and threshold-shifting ordinary channels. It does not close the full regularity problem, but it narrows the ordinary-channel burden needed before residual pathological refinement, coefficient recovery, and final bridge assembly.

1 Introduction

The three-dimensional incompressible Navier–Stokes equations are

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \qquad \nabla \cdot u = 0,$$

posed here on the periodic domain

$$\Omega = \mathbb{T}^3.$$

The vorticity is

$$\omega = \nabla \times u,$$

and satisfies

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega.$$

The nonlinear term

$$(\omega \cdot \nabla)u$$

is vortex stretching. It is the central three-dimensional amplification mechanism in the Navier–Stokes regularity problem.

For smooth solutions, define the enstrophy by

$$E_\omega(t) = \frac{1}{2} \int_\Omega |\omega|^2 \, \mathrm{d}V.$$

The classical enstrophy balance is

$$\frac{\mathrm{d}E_\omega}{\mathrm{d}t} = P(t) - D(t),$$

where

$$P(t) = \int_\Omega \omega_i S_{ij} \omega_j \, \mathrm{d}V$$

is vortex stretching,

$$S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

is the strain tensor, and

$$D(t) = \nu \int_\Omega |\nabla \omega|^2 \, \mathrm{d}V$$

is viscous enstrophy dissipation.

The formulation used here is classical. Standard references for the Navier–Stokes regularity problem, vorticity methods, incompressible flow, and enstrophy-based analysis include Fefferman [1], Ladyzhenskaya [2], Doering and Gibbon [6], and Majda and Bertozzi [7]. The geometric role of vorticity direction and vortex stretching is closely related to the work of Constantin and Fefferman [3] and Constantin [4].

Where $|\omega| > 0$, write

$$\omega = |\omega| n, \qquad n = \frac{\omega}{|\omega|}.$$

The local strain-alignment factor is

$$a(x,t) = n_i S_{ij} n_j,$$

and its positive part is

$$a^+(x,t) = \max\{a(x,t), 0\}.$$

The positive stretching density is

$$|\omega|^2 a^+(x,t).$$

Thus dangerous amplification requires more than large vorticity. It requires vorticity arranged so that positive strain alignment contributes to enstrophy growth.

The geometric cost available in the enstrophy balance is organized by the standard decomposition

$$|\nabla\omega|^2 = |\nabla|\omega||^2 + |\omega|^2 |\nabla n|^2.$$

The first term is magnitude-gradient cost. The second term is directional-gradient cost. The high-vorticity pinching program studies whether dangerous stretching must eventually pay one of these costs, lose alignment, fragment, become scale-local, enter a threshold complement, or become residual pathological concentration.

The 150-series separates this program into modular layers. Paper 150J [26] assembled the conditional enstrophy-closure theorem. Paper 150K [27] proved the unconditional accounting lemmas needed after a channel is visible and estimated, including partition-of-unity splitting, bounded-overlap accounting, pointwise and integrated Gronwall closure, burst summability, threshold splitting, and pathological-reduction bookkeeping. Paper 150L [28] formulated the refined visibility bridge: dangerous amplification must either enter the primary depletion regime or activate a named channel. Paper 150M [29] began the ordinary-channel control layer by studying the two most coherent ordinary channels,

$$R^+_{\text{patch}} \qquad \text{and} \qquad R_{\text{trans}}.$$

Paper 150M [29] showed that coherent aligned patches and transition layers should not remain undefined obstructions. They must either become absorbable, remain short-lived relative to the local stretching time, lose alignment, leak robustly, fragment, become scale-local, enter the threshold complement, or become residual pathology. In schematic form, Paper 150M handled the coherent routes:

$$R^+_{\text{patch}} + R_{\text{trans}} \quad \Longrightarrow \quad \text{cost, finite lifetime, or named exit channel.}$$

The present paper, Paper 150N, studies the ordinary exit channels that remain after coherent-channel control:

$$R_{\text{frag}}, \qquad R_{\text{scale}}, \qquad R_{\text{low}}.$$

These are the broken, filtered, and threshold-shifting ordinary channels.

The fragmentation channel R_{frag} is activated when positive stretching is distributed across many separated, semi-separated, moving, reconnecting, or intermittently active components. Fragmentation may help control stretching by creating interface cost, reducing coherent strain alignment, increasing component separation, shortening component lifetimes, or forcing many small supports to pay gradient or boundary costs. However, fragmentation is not automatically harmless. Many fragments may collectively preserve positive stretching. Therefore, fragmentation requires its own control target.

The scale-local channel R_{scale} is activated when dangerous stretching is not visible at the full-field level but becomes visible under filtering or scale decomposition. A route may look invisible only because the diagnostic scale is too coarse. Conversely, if a declared scale family is sufficiently complete and the dangerous contribution still avoids control, the route may become residual pathology. Thus scale-local control must distinguish resolved scale-local visibility from unresolved diagnostic failure and genuine scale-evading concentration. It must also keep the scale budget finite: a scale family that is too sparse can miss the active support, while a scale family with uncontrolled overlap can consume the dissipation margin.

The low-vorticity complement channel R_{low} is activated when positive stretching lies outside the selected high-vorticity region, near a threshold boundary, or in the complement of a smooth high-vorticity weight. This channel prevents threshold choices from hiding dangerous stretching. If stretching slips across a cutoff or flickers repeatedly through a threshold band, the contribution must still be counted. Smooth threshold weights, finite-budget threshold-flicker accounting, and complement estimates are therefore part of the ordinary-channel control problem.

The combined remaining ordinary-channel contribution is

$$R_{\text{rem,ord}}(t) = R_{\text{frag}}(t) + R_{\text{scale}}(t) + R_{\text{low}}(t).$$

The desired pointwise control target is

$$R_{\text{rem,ord}}(t) \le \delta_{\text{rem,ord}} D(t) + C_{\text{rem,ord}} E_\omega(t),$$

where

$$\delta_{\text{rem,ord}} = \delta_{\text{frag}} + \delta_{\text{scale}} + \delta_{\text{low}}.$$

For moving, intermittent, scale-dependent, reconnecting, or threshold-flickering structures, the pointwise estimate may be too strong. In that case, the target is a subinterval-stable integrated estimate:

$$\int_{t_0}^{t} R_{\text{rem,ord}}(s)\ \mathrm{d}s \le \delta_{\text{rem,ord}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\text{rem,ord}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_0$$

for every

$$t \in I = [t_0, t_1].$$

The phrase "for every $t \in I$" is essential. Paper 150K [27] showed that full-interval integrated estimates are not enough if they can hide intermediate spikes. Any integrated estimate used in the final assembly must be stable on subintervals or on stopping-time partial sums that control all intermediate times.

The margin requirement remains central. Paper 150J [26] closes only if the total dissipation coefficient remains below one. After Paper 150M, the coherent-channel coefficient is

$$\delta_{\text{coh}} = \delta_{\text{patch}} + \delta_{\text{trans}}.$$

After the present paper, the remaining ordinary-channel coefficient is

$$\delta_{\text{rem,ord}} = \delta_{\text{frag}} + \delta_{\text{scale}} + \delta_{\text{low}}.$$

If the pathological channel consumes coefficient δ_{path}, then the final margin condition is

$$\theta + \delta_{\text{coh}} + \delta_{\text{rem,ord}} + \delta_{\text{path}} < 1.$$

Thus Paper 150N is not merely about bounding $R_{\text{frag}}, R_{\text{scale}}, R_{\text{low}}$. It is about bounding them sharply enough that pathological refinement and final coefficient recovery remain possible.

The central thesis of Paper 150N is

$$\text{ordinary non-coherent exits} \quad \Longrightarrow \quad \text{cost, lower-order behavior, or pathological residual.}$$

Fragmentation should produce interface cost, reconnection cost, separation cost, loss of coherent alignment, lower-order distribution, finite lifetime, or residual pathology. Scale-local visibility should be captured by a declared scale family with controlled overlap and finite scale budget, absorbed by gradient or transfer cost, or become a scale-evading residual. Complement stretching should be controlled by threshold splitting, smooth weights, low-vorticity/lower-order estimates, finite-budget flicker control, or residual threshold pathology.

This paper is theoretical and structural. It does not introduce a new numerical model. Numerical diagnostics from earlier 150-series work may motivate which quantities should be tracked, but Paper 150N does not infer a theorem from simulations. The relevant quantities remain classical Navier–Stokes objects: vorticity, strain alignment, positive stretching, enstrophy, dissipation, component structure, reconnection neighborhoods, scale-filtered support, threshold weights, and time-integrated channel contributions.

Position in the 150-series. Paper 150N picks up exactly where Paper 150M stops. Paper 150M handled coherent structure: aligned patches and transition layers. Paper 150N handles what happens when coherent structure breaks, hides by scale, or slips through thresholds. Later work should then revisit residual pathology, recover sharp coefficients, and assemble the full bridge:

150M: coherent structure,

150N: broken, filtered, and threshold-shifting ordinary structure,

150O: residual pathological refinement,

150P: coefficient sharpness,

150Q: final bridge assembly.

The paper is organized as follows. Section 2 fixes the classical notation and the remaining ordinary-channel targets. Section 3 defines fragmentation geometry. Section 4 formulates fragmentation control targets. Section 5 introduces scale-local geometry and filtering. Section 6 formulates scale-local control targets. Section 7 studies low-vorticity complement and threshold-flicker geometry. Section 8 formulates complement control targets. Section 9 states the conditional remaining-ordinary-channel control theorem. Section 10 gives falsifiers and failure modes. Section 11 explains the relation to Papers 150J, 150K, 150L, and 150M. Section 12 concludes.

2 Classical Setup and Remaining Ordinary Channels

This section fixes the notation used throughout Paper 150N and states the remaining ordinary-channel control targets. The paper remains entirely within the classical three-dimensional incompressible Navier–Stokes framework. Its role is not to introduce a new fluid model, but to organize the ordinary downstream channels left after coherent aligned-patch and transition-layer control.

The channels studied in this paper are

$$R_{\text{frag}}, \qquad R_{\text{scale}}, \qquad R_{\text{low}}.$$

They represent fragmentation, scale-local visibility, and low-vorticity complement stretching. Together, they form the remaining ordinary-channel layer after Paper 150M [29] handles the coherent channels

$$R^{+}_{\text{patch}}, \qquad R_{\text{trans}}.$$

2.1 Navier–Stokes and vorticity form

Let

$$u : \Omega \times [0, T] \to \mathbb{R}^3$$

be a smooth divergence-free velocity field on the periodic domain

$$\Omega = \mathbb{T}^3.$$

The incompressible Navier–Stokes equations are

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \qquad \nabla \cdot u = 0,$$

where

$$\nu > 0$$

is the kinematic viscosity.

The vorticity is

$$\omega = \nabla \times u.$$

Taking the curl of the Navier–Stokes equations gives

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega.$$

The term

$$(\omega \cdot \nabla)u$$

is vortex stretching.

Writing the strain tensor as

$$S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i),$$

the total stretching production is

$$P(t) = \int_{\Omega} \omega_i S_{ij} \omega_j \, \mathrm{d}V.$$

2.2 Enstrophy balance

The enstrophy is

$$E_\omega(t) = \frac{1}{2}\int_\Omega |\omega(x,t)|^2 \, \mathrm{d}V.$$

The viscous enstrophy dissipation is

$$D(t) = \nu \int_\Omega |\nabla\omega(x,t)|^2 \, \mathrm{d}V.$$

For smooth periodic solutions, the classical enstrophy balance is

$$\frac{\mathrm{d}E_\omega}{\mathrm{d}t} = P(t) - D(t).$$

This identity is the closure mechanism used throughout the 150-series. The purpose of the channel framework is to show that every dangerous contribution to stretching is either absorbed by dissipation, reduced to lower-order enstrophy control, reclassified into a named channel, or isolated as residual pathology.

2.3 Positive stretching

Where

$$|\omega| > 0,$$

write

$$\omega = |\omega| n, \qquad n = \frac{\omega}{|\omega|}.$$

The local strain-alignment factor is

$$a(x,t) = n_i S_{ij} n_j.$$

The positive part is

$$a^+(x,t) = \max\{a(x,t), 0\}.$$

The positive stretching density is

$$|\omega|^2 a^+(x,t).$$

The total positive stretching reservoir is

$$P^+(t) = \int_\Omega |\omega|^2 a^+(x,t) \, \mathrm{d}V.$$

The remaining ordinary channels studied in this paper are all ways that positive stretching may persist after coherent-channel control has failed or exited downstream. Fragmentation distributes positive stretching across multiple pieces. Scale-local visibility hides positive stretching from full-field diagnostics until filtering is applied. Complement stretching places positive stretching outside the selected high-vorticity mask or inside a threshold transition band.

2.4 Gradient-cost decomposition

The standard decomposition of the vorticity-gradient density is

$$|\nabla\omega|^2 = |\nabla|\omega||^2 + |\omega|^2|\nabla n|^2.$$

Therefore,

$$D(t) = \nu \int_\Omega |\nabla|\omega||^2 \, \mathrm{d}V + \nu \int_\Omega |\omega|^2|\nabla n|^2 \, \mathrm{d}V.$$

The first term is magnitude-gradient cost. The second term is directional-gradient cost. Fragmentation may create interface, reconnection, or separation cost through both terms. Scale-local concentration may create high-frequency gradient cost. Threshold or complement behavior may force positive stretching into lower-vorticity regions where it can be controlled by enstrophy, smooth cutoff estimates, or lower-order bounds.

2.5 Primary estimate and channel remainder

The primary depletion architecture has schematic form

$$P(t) \le \theta D(t) + CE_\omega(t) + R_\kappa(t), \qquad 0 \le \theta < 1.$$

Here $\theta D(t)$ is the dissipation portion consumed by the primary estimate, $CE_\omega(t)$ is lower-order enstrophy control, and $R_\kappa(t)$ is the channel remainder.

The remainder is decomposed as

$$R_\kappa = R^+_{\mathrm{patch}} + R_{\mathrm{trans}} + R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}} + R_{\mathrm{path}}.$$

Paper 150M [29] studied the coherent contribution

$$R_{\mathrm{coh}} = R^+_{\mathrm{patch}} + R_{\mathrm{trans}}.$$

The present paper studies the remaining ordinary contribution

$$R_{\mathrm{rem,ord}} = R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}}.$$

The pathological channel

$$R_{\mathrm{path}}$$

is not controlled in this paper except as a residual exit route when an ordinary non-coherent channel fails to become absorbable or lower-order.

2.6 Fragmentation channel

The fragmentation channel

$$R_{\mathrm{frag}}$$

is activated when positive stretching is carried by many separated, semi-separated, moving, reconnecting, or intermittently active components. A schematic fragmentation support has the form

$$A(t) = \bigcup_j A_j(t),$$

where each component $A_j(t)$ carries some portion of the positive stretching density.

If $\pi_j^+(t)$ denotes the fraction of positive stretching carried by component j, a useful effective positive component count is

$$N_{\text{eff}}^+(t) = \left(\sum_j (\pi_j^+(t))^2\right)^{-1}.$$

Large or increasing $N_{\text{eff}}^+(t)$ indicates that stretching is distributed across many pieces.

Fragmentation is potentially favorable because separated components may pay interface or reconnection cost, lose coherent strain alignment, increase boundary cost, become short-lived, or become lower-order. However, fragmentation is not automatically controlled. Many fragments may collectively preserve positive stretching. Therefore, the channel requires a control estimate, a finite-budget event accounting, or a named exit.

A schematic fragmentation contribution is

$$R_{\text{frag}}(t) \sim \sum_j \int_{A_j(t)} |\omega|^2 a^+(x,t)\, \mathrm{d}V.$$

The desired control target is

$$R_{\text{frag}}(t) \le \delta_{\text{frag}} D(t) + C_{\text{frag}} E_\omega(t),$$

or a subinterval-stable integrated analogue.

2.7 Scale-local channel

The scale-local channel

$$R_{\text{scale}}$$

is activated when dangerous positive stretching is not visible at the full-field level but becomes visible after filtering or scale decomposition.

Let

$$\mathcal{L} = \{\ell_1, \ell_2, \dots, \ell_N\}$$

be a declared scale family, and let G_ℓ be a filter at scale ℓ. Define

$$u_\ell = G_\ell * u, \qquad \omega_\ell = \nabla \times u_\ell,$$

and

$$S_\ell = \frac{1}{2}\left(\nabla u_\ell + \nabla u_\ell^T\right).$$

A scale-local route is active when positive stretching becomes organized or significant at one or more declared scales.

A schematic scale-local contribution is

$$R_{\text{scale}}(t) \sim \sum_{\ell \in \mathcal{L}} \int_\Omega \chi_\ell(x,t)\, |\omega_\ell|^2 a_\ell^+(x,t)\, \mathrm{d}V,$$

where χ_ℓ is a scale-local channel weight and

$$a_\ell^+(x,t) = \max\{n_{\ell,i} S_{\ell,ij} n_{\ell,j}, 0\}.$$

Scale-local control must distinguish three cases:

(i) the route is resolved by the declared scale family and activates R_{scale};

(ii) the route is missed because the declared scale family is too coarse, which is diagnostic incompleteness rather than theorem-level control;

(iii) the route remains dangerous across a sufficiently complete declared scale family, in which case it becomes a scale-evading residual candidate for R_{path}.

The scale family is part of the theorem data. It must be fine enough to detect the active scale-local support and disciplined enough not to inflate the dissipation budget by uncontrolled overlap. The desired control target is

$$R_{\mathrm{scale}}(t) \le \delta_{\mathrm{scale}} D(t) + C_{\mathrm{scale}} E_\omega(t),$$

or a subinterval-stable integrated analogue.

2.8 Low-vorticity complement channel

The low-vorticity complement channel

$$R_{\mathrm{low}}$$

is activated when positive stretching lies outside the selected high-vorticity region, near a threshold boundary, or in the complement of a smooth high-vorticity weight.

For a sharp threshold, define

$$\Omega_\kappa(t) = \{x \in \Omega : |\omega(x,t)| > \kappa(t)\}.$$

The complement is

$$\Omega \setminus \Omega_\kappa(t).$$

A schematic low-vorticity complement contribution is

$$R_{\mathrm{low}}(t) \sim \int_{\Omega \setminus \Omega_\kappa(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V.$$

For a smooth high-vorticity weight $W_\kappa(|\omega|)$, the complement weight is

$$1 - W_\kappa(|\omega|),$$

and the exact threshold split is

$$W_\kappa(|\omega|) + (1 - W_\kappa(|\omega|)) = 1.$$

Applied to the positive stretching density, this gives

$$\int_\Omega |\omega|^2 a^+\ \mathrm{d}V = \int_\Omega W_\kappa(|\omega|)|\omega|^2 a^+\ \mathrm{d}V + \int_\Omega (1 - W_\kappa(|\omega|))|\omega|^2 a^+\ \mathrm{d}V.$$

Thus thresholding cannot make positive stretching disappear. It can only assign it to the high-vorticity side, the complement side, or a transition band. The purpose of R_{low} is to prevent the proof from hiding dangerous stretching behind a cutoff.

The desired control target is

$$R_{\text{low}}(t) \le \delta_{\text{low}} D(t) + C_{\text{low}} E_\omega(t),$$

or a subinterval-stable integrated analogue. If repeated threshold crossing occurs, the associated flicker constants must be summable or controlled by a finite budget.

2.9 Combined remaining ordinary-channel target

The combined remaining ordinary-channel contribution is

$$R_{\text{rem,ord}}(t) = R_{\text{frag}}(t) + R_{\text{scale}}(t) + R_{\text{low}}(t).$$

The ideal pointwise estimate is

$$R_{\text{rem,ord}}(t) \le \delta_{\text{rem,ord}} D(t) + C_{\text{rem,ord}} E_\omega(t),$$

where

$$\delta_{\text{rem,ord}} = \delta_{\text{frag}} + \delta_{\text{scale}} + \delta_{\text{low}},$$

and

$$C_{\text{rem,ord}} = C_{\text{frag}} + C_{\text{scale}} + C_{\text{low}}.$$

For moving, intermittent, reconnecting, scale-dependent, or threshold-flickering structures, the appropriate target may instead be

$$\int_{t_0}^{t} R_{\text{rem,ord}}(s)\ \mathrm{d}s \le \delta_{\text{rem,ord}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\text{rem,ord}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_0$$

for every

$$t \in I = [t_0, t_1].$$

This subinterval-stability requirement is inherited from Paper 150K [27]. It prevents a full-interval integrated estimate from hiding an intermediate enstrophy spike.

2.10 Meaning of lower-order control

Throughout this paper, a channel contribution is called lower-order if it can be bounded without consuming an uncontrolled amount of the dissipation margin. The basic form is

$$R_j(t) \le C_j E_\omega(t),$$

or, more generally,

$$R_j(t) \le \delta_j D(t) + C_j E_\omega(t)$$

with δ_j small enough to preserve the final margin.

In integrated form, lower-order control means

$$\int_{t_0}^{t} R_j(s)\,\mathrm{d}s \le \delta_j \int_{t_0}^{t} D(s)\,\mathrm{d}s + C_j \int_{t_0}^{t} E_\omega(s)\,\mathrm{d}s + C_{0,j}$$

for every $t \in I$, with $C_{0,j}$ finite or summable under the declared stopping-time decomposition.

Thus "lower-order" is not a qualitative label. It means that the contribution can be inserted into the Paper 150J [26] enstrophy closure without exhausting the dissipation budget.

2.11 Margin requirement

The remaining ordinary-channel estimate is useful for the Paper 150J [26] assembly only if it preserves the final dissipation margin. The primary estimate consumes coefficient θ. The coherent channels from Paper 150M [29] consume

$$\delta_{\mathrm{coh}} = \delta_{\mathrm{patch}} + \delta_{\mathrm{trans}}.$$

The remaining ordinary channels consume

$$\delta_{\mathrm{rem,ord}} = \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}}.$$

The pathological channel, if not reduced, consumes coefficient

$$\delta_{\mathrm{path}}.$$

The final margin condition is

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1.$$

If R_{path} reduces completely to ordinary channels, then $\delta_{\mathrm{path}} = 0$ as an independent coefficient, and the transferred contribution must be included in the appropriate ordinary-channel budget.

Thus Paper 150N is not only a classification paper. It is a budget paper. Fragmentation, scale-local visibility, and complement stretching must be controlled with coefficients sharp enough to leave room for any residual pathological contribution and for final coefficient recovery.

2.12 Budget hierarchy for remaining ordinary channels

The coefficient

$$\delta_{\mathrm{rem,ord}} = \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}}$$

must be interpreted as a zero-sum dissipation budget. Each remaining ordinary channel consumes part of the same reserve left after primary depletion and coherent-channel control.

The intended hierarchy is:

$$\delta_{\mathrm{frag}} \quad \text{is expected to be interface-cost dominated,}$$

$$\delta_{\mathrm{scale}} \quad \text{is expected to be high-frequency or transfer-cost dominated,}$$

and

$$\delta_{\mathrm{low}} \quad \text{is expected to be lower-order or threshold-accounting dominated.}$$

This hierarchy is not a theorem. It records the intended bridge structure. Fragmentation should be paid for mainly by interface, reconnection, boundary, separation, lifetime, or alignment-loss costs. Scale-locality should be paid for mainly by high-frequency, transfer, finite-scale-budget, or scale-boundary costs. Complement stretching should be paid for mainly by lower-order enstrophy control, smooth-threshold splitting, finite threshold-band cost, or flicker summability.

The most dangerous budget failure is complement dominance:

$$\delta_{\mathrm{low}} \approx 1 - \theta - \delta_{\mathrm{coh}}.$$

If R_{low} consumes most of the available reserve, then little margin remains for scale-local transfer, fragmentation residue, or residual pathology. A successful remaining ordinary-channel estimate should therefore leave a declared downstream reserve:

$$\delta_{\mathrm{rem,ord}} \leq (1 - \theta - \delta_{\mathrm{coh}}) - \delta_{\mathrm{reserve,N}}, \qquad \delta_{\mathrm{reserve,N}} > 0.$$

This reserve is needed for R_{path} and final coefficient recovery. Paper 150N therefore asks not only whether R_{frag}, R_{scale}, and R_{low} can be bounded, but whether they can be bounded cheaply enough to keep the final Paper 150J margin alive.

2.13 Channel exits and residual pathology

Each remaining ordinary channel has possible exit routes.

Fragmentation may become lower-order, pay interface or reconnection cost, lose coherent alignment, become scale-local, enter the complement, or become residual pathology:

$$R_{\mathrm{frag}} \Longrightarrow \text{cost, lower-order behavior, } R_{\mathrm{scale}},\ R_{\mathrm{low}},\ \text{or } R_{\mathrm{path}}.$$

Scale-local support may be captured by the declared scale family, pay gradient or transfer cost, move into the complement, or become scale-evading pathology:

$$R_{\mathrm{scale}} \Longrightarrow \text{resolved scale cost, } R_{\mathrm{low}},\ \text{or } R_{\mathrm{path}}.$$

Complement stretching may be controlled by smooth threshold splitting, lower-order estimates, threshold-band accounting, finite-budget flicker control, or become residual threshold pathology:

$$R_{\mathrm{low}} \Longrightarrow \text{threshold control, lower-order behavior, or } R_{\mathrm{path}}.$$

These exits do not close the final theorem by themselves. They identify the remaining dependency. If a route exits to R_{path}, then it becomes part of the residual pathological-channel problem, not an ordinary-channel success.

2.14 No double-counting

The channel decomposition is useful only if the same positive-stretching contribution is not counted multiple times. A fragmented route may also be scale-local. A scale-local route may cross a threshold. A complement route may appear through many fragmented components. These overlaps are expected.

Paper 150K [27] supplies the accounting rule. A valid decomposition must use one of the following:

(i) a measurable partition of the stretching-active support;

(ii) a partition of unity over channel weights;

(iii) a bounded-overlap family with the overlap constant charged to the margin;

(iv) a stopping-time decomposition;

(v) a scale decomposition with declared overlap control;

(vi) or an equivalent bookkeeping device.

If channel weights $\chi_j(x,t)$ form an exact partition,

$$\sum_j \chi_j(x,t) = 1,$$

then positive stretching is assigned once. If instead

$$\sum_j \chi_j(x,t) \le K_{\mathrm{ov}},$$

then the overlap factor K_{ov} must be included in the dissipation budget.

Thus channel overlap is allowed only when it is paid for explicitly. The final margin cannot be preserved by silently counting the same stretching contribution in several channels.

2.15 What this paper proves and does not prove

Paper 150N formulates the control targets for

$$R_{\mathrm{frag}}, \qquad R_{\mathrm{scale}}, \qquad R_{\mathrm{low}}.$$

It studies how fragmentation, scale-local visibility, and threshold/complement behavior may become absorbable, lower-order, or residual pathological.

It does not prove full Navier–Stokes regularity. It does not control the coherent channels already studied in Paper 150M [29] except through references to their downstream exits. It does not control R_{path} except by identifying when an ordinary route exits to residual pathology. It does not recover final coefficient sharpness. Those are downstream bridge tasks.

The role of Paper 150N is narrower:

$$\text{broken, filtered, and threshold-shifting ordinary channels} \Longrightarrow \text{control target, lower-order route, or named obstruc}$$

2.16 Summary

This section fixed the classical Navier–Stokes notation, the enstrophy balance, the positive-stretching density, the gradient-cost decomposition, and the remaining ordinary-channel targets:

$$R_{\mathrm{frag}}, \qquad R_{\mathrm{scale}}, \qquad R_{\mathrm{low}}.$$

The combined target is

$$R_{\text{frag}} + R_{\text{scale}} + R_{\text{low}} \le \delta_{\text{rem,ord}} D + C_{\text{rem,ord}} E_\omega,$$

or a subinterval-stable integrated analogue.

The final Paper 150J [26] margin requires

$$\theta + \delta_{\text{coh}} + \delta_{\text{rem,ord}} + \delta_{\text{path}} < 1.$$

Thus the remaining ordinary channels must be controlled sharply enough to preserve reserve for residual pathology and final coefficient recovery. The next section defines fragmentation geometry.

3 Fragmentation Geometry

The previous section fixed the remaining ordinary-channel targets. This section defines the first of those targets:

$$R_{\text{frag}}.$$

The fragmentation channel is activated when dangerous positive stretching is no longer carried by one coherent aligned support or one protected transition-layer core, but instead is distributed across many separated, semi-separated, moving, reconnecting, or intermittently active components.

Fragmentation is not automatically favorable and not automatically dangerous. It may help control amplification by breaking coherent strain alignment, creating interface cost, increasing boundary cost, shortening component lifetimes, or spreading positive stretching into lower-order pieces. But fragmentation may also preserve amplification if many components remain collectively aligned with positive strain. The purpose of this section is to define the geometry of that channel before its control targets are stated in Section 4.

3.1 Fragmented stretching-active support

Let

$$A(t) \subset \Omega$$

be a stretching-active support. In the fragmentation channel, this support decomposes into components

$$A(t) = \bigcup_{j \in J(t)} A_j(t),$$

where $J(t)$ is a finite or countable index set, and each $A_j(t)$ is a separated or semi-separated stretching-active component.

The positive stretching carried by component $A_j(t)$ is

$$P_j^+(t) = \int_{A_j(t)} |\omega|^2 a^+(x,t)\, \mathrm{d}V.$$

The total positive stretching carried by the fragmented support is

$$P_A^+(t) = \sum_{j \in J(t)} P_j^+(t) = \sum_{j \in J(t)} \int_{A_j(t)} |\omega|^2 a^+(x,t)\, \mathrm{d}V.$$

A fragmented route is relevant only if

$$P_A^+(t)$$

or its time integral over the interval under consideration is non-negligible relative to the channel budget.

The fragmentation channel is therefore not defined merely by the number of components. It is defined by the distribution of positive stretching across those components.

3.2 Positive-stretching fractions

For each component, define the positive-stretching fraction

$$\pi_j^+(t) = \frac{P_j^+(t)}{\sum_{k \in J(t)} P_k^+(t) + \varepsilon},$$

where $\varepsilon > 0$ is a small regularization constant used only to avoid division by zero in diagnostics.

The fractions satisfy, up to the regularization,

$$\sum_{j \in J(t)} \pi_j^+(t) \approx 1$$

when the fragmented support carries nonzero positive stretching.

A single dominant component gives

$$\max_j \pi_j^+(t) \approx 1.$$

A genuinely fragmented positive-stretching reservoir gives a distribution in which several or many components carry non-negligible fractions. Thus fragmentation should be measured by positive-stretching distribution, not only by geometric component count.

3.3 Effective positive component count

A useful diagnostic for fragmentation is the effective positive component count

$$N_{\text{eff}}^+(t) = \left(\sum_{j \in J(t)} \left(\pi_j^+(t)\right)^2 \right)^{-1}.$$

This quantity is small when one component dominates the positive stretching and large when the positive stretching is spread across many components.

If m components carry equal positive-stretching fractions, then

$$\pi_j^+(t) \approx \frac{1}{m},$$

and therefore

$$N_{\text{eff}}^+(t) \approx m.$$

Thus $N_{\text{eff}}^+(t)$ measures the effective number of stretching-active components rather than the raw number of geometric pieces.

This distinction matters. A flow may contain many small geometric components, but if one component carries almost all positive stretching, the route is closer to an aligned-patch or transition-layer obstruction than to a true fragmentation channel. Conversely, a modest number of components may already be fragmentation-relevant if they share positive stretching comparably.

3.4 Fragmentation versus aligned-patch support

Fragmentation differs from coherent aligned-patch support.

The aligned-patch channel

$$R^+_{\text{patch}}$$

is active when positive stretching is carried by coherent support on which vorticity direction remains organized and favorably aligned with strain. A coherent aligned patch may be compact, broad, moving, or weighted, but its defining feature is coherent support.

The fragmentation channel

$$R_{\text{frag}}$$

is active when the support carrying positive stretching is no longer one coherent patch but has separated into multiple stretching-active pieces:

$$\Omega_{\text{patch}}(t) \Longrightarrow \bigcup_j A_j(t).$$

This transition can be written schematically as

$$R^+_{\text{patch}} \Longrightarrow R_{\text{frag}}.$$

It is not automatically control. It is reclassification. The same positive-stretching contribution should be reassigned from the aligned-patch channel to the fragmentation channel using a partition, stopping-time reassignment, or bounded-overlap rule. The contribution should not be counted simultaneously as both R^+_{patch} and R_{frag} unless an explicit overlap cost is charged.

3.5 Fragmentation versus transition-layer support

Fragmentation also differs from transition-layer support.

The transition-layer channel

$$R_{\text{trans}}$$

is active when a stretching-active core is protected by a surrounding transition region:

$$\Omega_{\text{core}}(t) \cup \Omega_{\text{trans}}(t).$$

The core-layer structure remains organized even if the boundary is dynamically important.

The fragmentation channel becomes active when that organized core-layer picture breaks into multiple stretching-active pieces:

$$\Omega_{\text{core}}(t) \cup \Omega_{\text{trans}}(t) \Longrightarrow \bigcup_j A_j(t).$$

If the fragments collectively carry significant positive stretching, then the route exits from transition-layer control to fragmentation control:

$$R_{\text{trans}} \Longrightarrow R_{\text{frag}}.$$

As with aligned-patch exits, this is a channel transition, not a proof of absorption. The burden shifts from coherent-channel control to fragmentation control.

3.6 Geometric separation

Fragmentation requires some notion of separation. Components may be separated by distance, by weak positive stretching between them, by loss of vorticity-direction coherence, by scale-local filtering, or by a stopping-time/component-tracking rule.

A hard geometric separation condition may take the form

$$\operatorname{dist}(A_i(t), A_j(t)) \geq r_{\text{sep}}(t) \qquad \text{for } i \neq j.$$

However, hard separation is not always the right diagnostic. In a smooth flow, components may be connected by weak bridges or diffuse transition regions. A weighted or graph-based description may be more robust.

One may introduce component weights

$$\chi_j(x,t) \geq 0$$

such that

$$\chi_j$$

is concentrated near component $A_j(t)$. A bounded-overlap condition

$$\sum_j \chi_j(x,t) \leq K_{\text{frag}}$$

then allows components to be separated in an accounting sense, even when their geometric boundaries are not sharp. If $K_{\text{frag}} > 1$, the overlap must be paid for in the dissipation margin, following the accounting rule of Paper 150K [27].

3.7 Interface and boundary geometry

Fragmentation often creates interface or boundary cost. If the components $A_j(t)$ are separated by boundaries, transition regions, weak bridges, or reconnection neighborhoods, then one expects magnitude-gradient or directional-gradient cost near those interfaces.

A schematic interface neighborhood is

$$N_r(\partial A_j(t)),$$

where $r > 0$ is a declared interface thickness. A schematic fragmentation interface cost is

$$C_{\text{int}}(t) = \nu \sum_{j \in J(t)} \int_{N_r(\partial A_j(t))} |\nabla \omega|^2 \, \mathrm{d}V.$$

Using the gradient decomposition,

$$|\nabla\omega|^2 = |\nabla|\omega||^2 + |\omega|^2|\nabla n|^2,$$

the interface cost separates into magnitude-gradient and directional-gradient contributions:

$$C_{\mathrm{int}}(t) = \nu \sum_j \int_{N_r(\partial A_j(t))} |\nabla|\omega||^2 \ \mathrm{d}V + \nu \sum_j \int_{N_r(\partial A_j(t))} |\omega|^2|\nabla n|^2 \ \mathrm{d}V.$$

A fragmentation route is favorable for control if significant positive stretching forces enough interface cost. The difficult case is fragmentation that preserves positive stretching while paying too little interface or boundary cost.

3.8 Alignment loss under fragmentation

Fragmentation may reduce dangerous amplification by breaking coherent strain alignment. The positive stretching carried by component $A_j(t)$ depends on

$$a^+(x,t) = \max\{n_i S_{ij} n_j, 0\}.$$

If components become misaligned with the strain field, then their positive-stretching contribution decreases even if their vorticity magnitude remains high.

A component-level aligned fraction may be defined by

$$\Pi_j^+(t) = \frac{\int_{A_j(t)} |\omega|^2 a^+(x,t) \ \mathrm{d}V}{\int_{A_j(t)} |\omega|^2 |a(x,t)| \ \mathrm{d}V + \varepsilon}.$$

A fragmentation route becomes less dangerous if many components have small $\Pi_j^+(t)$. It remains dangerous if many components preserve strong positive alignment:

$$\Pi_j^+(t) \approx 1$$

for a significant fraction of the positive-stretching reservoir.

Thus fragmentation control should not rely only on component count. It must track whether the fragments remain positively aligned.

3.9 Fragmentation and component lifetimes

Fragmented components may be short-lived. A component $A_j(t)$ is dynamically dangerous only if it persists long enough to contribute meaningful integrated positive stretching.

Let

$$I_j = [\tau_j, \tau_{j+1}]$$

be a stopping-time interval on which component $A_j(t)$ is trackable. The integrated positive stretching carried by that component is

$$\int_{I_j} \int_{A_j(t)} |\omega|^2 a^+(x,t) \ \mathrm{d}V \ \mathrm{d}t.$$

Short-lived fragments may be lower-order if this integral is small. Long-lived fragments, or repeatedly reappearing fragments, may remain dangerous.

A fragmentation estimate may therefore be pointwise or integrated. If integrated, it must be stable under subintervals or stopping-time partial sums:

$$\sum_{I_j \subset [t_0,t]} C_{0,j} \leq C_0$$

for all $t \in I$, if component-level residual constants $C_{0,j}$ are used. This prevents a sequence of many small fragments from producing a nonsummable burst contribution.

3.10 Moving and reconnecting fragments

Fragments may move, merge, split, or reconnect. A fixed spatial component decomposition may fail even when the fragmented route remains active. Therefore, fragmentation geometry should allow moving and stopping-time component descriptions.

A moving component family

$$A_j(t)$$

is acceptable if it is trackable by a material, weighted, graph-based, or stopping-time diagnostic. If components move too rapidly for fixed masks, one may use a time-dependent partition of unity

$$\chi_j(x,t)$$

or stopping-time intervals on which the component assignment is stable.

Reconnection requires special care. If two components merge, the positive-stretching contribution should be reassigned without double-counting. If one component splits, the new pieces should inherit the appropriate portion of the channel weight. These reassignment events are part of the fragmentation channel only if their accumulated cost and residual constants remain controlled.

If moving or reconnecting fragments cannot be tracked, and yet their cumulative positive stretching remains significant, the route may exit to residual pathology:

$$R_{\text{frag}} \Longrightarrow R_{\text{path}}.$$

3.11 Reconnection as an interface-cost event

Fragmentation is not only a static component count. In three-dimensional flow, fragmented supports may reconnect. A reconnection event can temporarily increase geometric complexity while preserving positive strain alignment. Such a route is dangerous if it allows many pieces to merge, braid, or form a necklace-like support without paying enough gradient cost.

Paper 150N treats reconnection in one of three ways.

First, if reconnection produces sharp changes in vorticity magnitude or vorticity direction near the joining region, then it is an interface-cost event. The relevant cost is included in

$$C_{\text{int}}(t) = \nu \sum_j \int_{N_r(\partial A_j(t))} |\nabla\omega|^2 \, dV,$$

or in an analogous reconnection-neighborhood cost

$$C_{\mathrm{rec}}(t) = \nu \int_{N_r(\mathcal{R}_{\mathrm{rec}}(t))} |\nabla\omega|^2 \, \mathrm{d}V,$$

where $\mathcal{R}_{\mathrm{rec}}(t)$ denotes the active reconnection region.

Second, if reconnection is smooth enough that no significant gradient cost appears, then the route must be tested for preserved coherent alignment. In that case, the contribution may exit back to a coherent channel, such as R^+_{patch} or R_{trans}, or to a scale-local channel if the reconnection is only visible after filtering.

Third, if reconnection preserves significant positive stretching while avoiding interface cost, coherent-channel classification, scale-local classification, complement assignment, and absorbability, then it is assigned to residual pathology:

$$R_{\mathrm{frag}} \Longrightarrow R_{\mathrm{path}}.$$

Thus reconnection is not an untracked loophole in the fragmentation channel. It must appear as interface cost, coherent reclassification, scale-local structure, complement behavior, or residual pathology.

3.12 Fragmentation and scale-locality

Fragmentation and scale-locality are closely related. A support may appear as many fragments at one scale but as a coherent structure at another. Conversely, a coherent-looking support may become fragmented after filtering or refinement.

Let

$$\mathcal{L} = \{\ell_1, \ell_2, \ldots, \ell_N\}$$

be a declared scale family. A component decomposition should specify the scale at which the components are identified. If the apparent fragmentation disappears under a nearby admissible scale, then the classification may be scale-sensitive.

A robust fragmentation claim should therefore state one of the following:

(i) fragmentation is stable across a declared range of scales;

(ii) the route exits to the scale-local channel R_{scale};

(iii) the scale family is too coarse, so the classification is diagnostic rather than theorem-level;

(iv) the route remains scale-evading and exits to R_{path}.

Thus fragmentation control and scale-local control are distinct but linked. Paper 150N treats both so that broken support and filtered support can be assigned without losing positive stretching.

3.13 Fragmentation and threshold dependence

Fragmented supports may also depend on high-vorticity thresholds. Let

$$\Omega_\kappa(t) = \{x \in \Omega : |\omega(x,t)| > \kappa(t)\}.$$

A fragmented set inside $\Omega_\kappa(t)$ may change component count when $\kappa(t)$ changes. Some fragments may lie partly outside the thresholded region. Others may flicker across the threshold boundary.

If significant positive stretching lies outside $\Omega_\kappa(t)$, then the route activates the low-vorticity complement channel:

$$R_{\mathrm{low}}.$$

If the component structure changes sharply under small threshold changes, then the route has threshold instability. A smooth high-vorticity weight $W_\kappa(|\omega|)$ may be needed to avoid cutoff artifacts.

Thus fragmentation should not hide complement stretching. A fragmented route near a threshold must be split into high-vorticity and complement contributions without loss:

$$W_\kappa(|\omega|) + \big(1 - W_\kappa(|\omega|)\big) = 1.$$

3.14 Fragmentation contribution

The fragmentation-channel contribution is the part of the remainder assigned to fragmented positive-stretching support:

$$R_{\mathrm{frag}}(t).$$

In a hard component decomposition, one may write schematically

$$R_{\mathrm{frag}}(t) = \sum_{j \in J(t)} \int_{A_j(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V.$$

In a weighted decomposition, one may write

$$R_{\mathrm{frag}}(t) = \sum_{j \in J(t)} \int_{\Omega} \chi_j(x,t) |\omega|^2 a^+(x,t)\ \mathrm{d}V,$$

with component weights $\chi_j \geq 0$.

If the component weights form an exact partition over the fragmented channel support, then the contribution is counted once. If they overlap, the overlap constant must be included in the budget. This is essential because a single region may appear simultaneously fragmented, scale-local, and threshold-sensitive unless the assignment rule is explicit.

3.15 Definition of fragmentation channel

We now give the working definition.

Definition 3.1 (Fragmentation channel) *Let u be a smooth solution of the three-dimensional incompressible Navier–Stokes equations on*

$$I = [t_0, t_1].$$

The fragmentation channel $R_{\rm frag}(I)$ is active on I if there exists a family of stretching-active supports

$$A(t) = \bigcup_{j \in J(t)} A_j(t)$$

or a weighted/stopping-time analogue such that:

(i) the fragmented support carries significant positive stretching,

$$\int_I \sum_{j \in J(t)} \int_{A_j(t)} |\omega|^2 a^+(x,t) \; \mathrm{d}V \; \mathrm{d}t$$

is non-negligible relative to the channel budget;

(ii) the positive stretching is distributed across multiple components, measured by a nontrivial effective component count $N^+_{\rm eff}(t)$, a stable weighted decomposition, or a stopping-time component family;

(iii) the support is no longer adequately described as one coherent aligned patch or one protected transition-layer core;

(iv) the components are separated, semi-separated, weighted, scale-localized, reconnecting, or stopping-time trackable in a declared diagnostic;

(v) the contribution is not already assigned entirely to primary entry, coherent aligned-patch support, transition-layer support, scale-local transfer, low-vorticity complement stretching, or residual pathology;

(vi) any overlap with other channel diagnostics is handled by a partition, bounded-overlap budget, stopping-time reassignment, or equivalent no-double-counting rule.

This definition makes fragmentation a positive support-geometry class, not a vague statement that a structure has become complicated. The channel is active when positive stretching is genuinely distributed across multiple stretching-active pieces.

3.16 Fragmentation exits

Once fragmentation is active, several outcomes are possible.

First, fragmentation may be directly absorbable:

$$R_{\rm frag}(t) \le \delta_{\rm frag} D(t) + C_{\rm frag} E_\omega(t).$$

Second, fragmentation may be integrably absorbable:

$$\int_{t_0}^t R_{\rm frag}(s) \; \mathrm{d}s \le \delta_{\rm frag} \int_{t_0}^t D(s) \; \mathrm{d}s + C_{\rm frag} \int_{t_0}^t E_\omega(s) \; \mathrm{d}s + C_{0,\rm frag}$$

for every $t \in I$.

Third, fragmentation may become lower-order if the component contributions lose positive alignment, become short-lived, or distribute stretching too weakly to overcome the available enstrophy budget.

Fourth, fragmentation may exit to the scale-local channel:

$$R_{\rm frag} \Longrightarrow R_{\rm scale},$$

if the relevant component geometry is meaningful only after filtering or scale decomposition.

Fifth, fragmentation may exit to the complement channel:

$$R_{\rm frag} \Longrightarrow R_{\rm low},$$

if significant stretching lies outside the selected high-vorticity mask or flickers through a threshold boundary.

Sixth, fragmentation may exit to residual pathology:

$$R_{\rm frag} \Longrightarrow R_{\rm path},$$

if positive stretching remains significant while fragmentation control, reconnection-cost accounting, scale-local classification, complement assignment, and absorbability all fail.

3.17 What fragmentation geometry does not prove

This section defines fragmentation geometry. It does not prove fragmentation control.

In particular, it does not prove that many components always pay enough interface cost. It does not prove that reconnecting components always pay enough reconnection cost. It does not prove that fragmentation always reduces positive strain alignment. It does not prove that all component bursts are summable. It does not prove that moving and reconnecting fragments are trackable. It does not prove that scale-sensitive fragmentation is controlled. It does not prove that threshold-sensitive fragmentation is lower-order.

Those are control questions. They are addressed as theorem targets in the next section.

3.18 Summary

The fragmentation channel $R_{\rm frag}$ is activated when significant positive stretching is distributed across multiple stretching-active components. The relevant diagnostics include positive-stretching fractions π_j^+, effective positive component count $N_{\rm eff}^+$, interface cost, reconnection cost, alignment loss, component lifetime, moving or reconnecting component tracking, scale dependence, and threshold dependence.

Fragmentation is an ordinary channel, not residual pathology, when the component structure is visible and classifiable. It becomes favorable for control if it creates interface or reconnection cost, weakens alignment, shortens lifetimes, becomes lower-order, or exits to a controlled scale or complement channel. It becomes pathological only if significant positive stretching survives fragmentation while all ordinary control and classification routes fail.

The next section formulates the control targets for $R_{\rm frag}$.

4 Fragmentation Control Targets

The previous section defined fragmentation geometry. This section states the control targets for the fragmentation channel

$$R_{\text{frag}}.$$

The goal is not to prove unconditional fragmentation control in this paper. The goal is to state what must be shown for fragmented positive-stretching support to become absorbable, lower-order, or a named downstream obstruction in the Paper 150J [26] assembly.

The central question is:

> If coherent support breaks into many pieces, does the broken structure pay enough cost?

Fragmentation is favorable only if it reduces coherent stretching, creates interface, reconnection, or boundary cost, shortens component lifetimes, pushes the route into scale-local or complement control, or isolates a residual pathological route. Fragmentation is not favorable if many components continue to preserve positive stretching while paying too little cost.

4.1 The fragmentation obstruction

A fragmented route is obstructive when significant positive stretching is distributed across many components:

$$A(t) = \bigcup_{j \in J(t)} A_j(t),$$

with

$$P_j^+(t) = \int_{A_j(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V,$$

and

$$P_A^+(t) = \sum_{j \in J(t)} P_j^+(t)$$

remaining non-negligible relative to the channel budget.

The obstruction is strongest when:

(i) $N_{\text{eff}}^+(t)$ is large or increasing;

(ii) many components retain positive strain alignment;

(iii) interface, reconnection, and boundary costs remain too small;

(iv) component lifetimes are long enough to contribute to integrated growth;

(v) the component structure is stable enough to preserve positive stretching but not stable enough to reduce to a coherent channel;

(vi) the route does not become scale-local, enter the complement, or become absorbable.

In schematic form, the dangerous regime is

$$\sum_j \int_{A_j(t)} |\omega|^2 a^+ \, \mathrm{d}V \quad \text{large},$$

while

$$\nu \sum_j \int_{N_r(\partial A_j(t))} |\nabla \omega|^2 \, \mathrm{d}V$$

and other available costs remain too small to absorb the contribution.

4.2 Pointwise fragmentation target

The strongest useful target is a pointwise estimate:

$$R_{\mathrm{frag}}(t) \leq \delta_{\mathrm{frag}} D(t) + C_{\mathrm{frag}} E_\omega(t).$$

Here $\delta_{\mathrm{frag}} \geq 0$ is the dissipation fraction consumed by the fragmentation channel, and $C_{\mathrm{frag}} \geq 0$ is a lower-order enstrophy coefficient.

This estimate is useful for closure only if the total margin remains positive:

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} < 1.$$

Equivalently,

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1, \qquad \delta_{\mathrm{rem,ord}} = \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}}.$$

Thus fragmentation control must be sharp enough not merely to bound R_{frag}, but to leave dissipation reserve for scale-local transfer, complement stretching, residual pathology, and final coefficient recovery.

4.3 Integrated fragmentation target

Fragmentation may be moving, intermittent, or component-changing. A pointwise estimate may be too strong if components split, merge, reconnect, or appear only on stopping-time intervals. In that case, the appropriate target is a subinterval-stable integrated estimate:

$$\int_{t_0}^{t} R_{\mathrm{frag}}(s) \, \mathrm{d}s \leq \delta_{\mathrm{frag}} \int_{t_0}^{t} D(s) \, \mathrm{d}s + C_{\mathrm{frag}} \int_{t_0}^{t} E_\omega(s) \, \mathrm{d}s + C_{0,\mathrm{frag}}$$

for every

$$t \in I = [t_0, t_1].$$

The phrase "for every $t \in I$" is essential. A full-interval estimate on $[t_0, t_1]$ alone may hide an intermediate enstrophy spike. Paper 150K [27] supplies the accounting principle: integrated estimates are useful for regularity only when they are stable on all subintervals or on stopping-time partial sums that control every intermediate time.

If fragmentation is controlled on burst intervals

$$I_m = [\tau_m, \tau_{m+1}],$$

then one needs

$$\sum_m C_{0,m} < \infty$$

or a finite-budget mechanism proving that the burst constants are summable. Otherwise, a Zeno-style sequence of fragmenting events can defeat the integrated estimate.

4.4 Interface-cost route

The most direct fragmentation-control mechanism is interface cost. If a stretching-active support breaks into many components, then boundaries, transition regions, weak bridges, or reconnection neighborhoods should appear between components.

A schematic interface cost is

$$C_{\mathrm{int}}(t) = \nu \sum_{j \in J(t)} \int_{N_r(\partial A_j(t))} |\nabla \omega|^2 \, \mathrm{d}V.$$

Using

$$|\nabla \omega|^2 = |\nabla |\omega||^2 + |\omega|^2 |\nabla n|^2,$$

this includes both magnitude-gradient and directional-gradient cost.

A possible interface-cost target is

$$R_{\mathrm{frag}}(t) \le c_{\mathrm{int}} C_{\mathrm{int}}(t) + C_{\mathrm{int},E} E_\omega(t),$$

with effective coefficient small enough to fit inside the remaining margin.

The difficult case is fragmentation without sufficient interface cost:

$$R_{\mathrm{frag}}(t) \text{ large}, \qquad C_{\mathrm{int}}(t) \text{ small}.$$

Such a route would show that many components can preserve positive stretching while avoiding the boundary, transition, or reconnection cost expected from fragmentation.

4.5 Reconnection-cost route

A fragmented route may undergo reconnection. During reconnection, components merge, braid, exchange support, or form necklace-like structures. Such events matter because they may preserve positive stretching while changing component topology.

A reconnection-control target is

$$R_{\mathrm{rec}}(t) \le \delta_{\mathrm{rec}} D(t) + C_{\mathrm{rec}} E_\omega(t),$$

where R_{rec} denotes the positive-stretching contribution carried by reconnection regions. Equivalently, one may require

$$R_{\mathrm{rec}}(t) \le c_{\mathrm{rec}} C_{\mathrm{rec}}(t) + C_{\mathrm{rec},E} E_\omega(t),$$

where

$$C_{\rm rec}(t) = \nu \int_{N_r(\mathcal{R}_{\rm rec}(t))} |\nabla\omega|^2 \, \mathrm{d}V.$$

If reconnection occurs on stopping-time intervals I_m, the integrated target is

$$\sum_m \int_{I_m} R_{\rm rec}(t) \, \mathrm{d}t \le \delta_{\rm rec} \sum_m \int_{I_m} D(t) \, \mathrm{d}t + C_{\rm rec} \sum_m \int_{I_m} E_\omega(t) \, \mathrm{d}t + \sum_m C_{0,m}^{\rm rec},$$

with

$$\sum_m C_{0,m}^{\rm rec} < \infty.$$

If reconnection preserves high positive stretching while producing neither interface cost nor summable integrated cost, then fragmentation control fails and the route must be assigned to $R_{\rm path}$.

4.6 Alignment-loss route

Fragmentation may help control amplification by breaking coherent positive strain alignment. For each component $A_j(t)$, define the aligned fraction

$$\Pi_j^+(t) = \frac{\int_{A_j(t)} |\omega|^2 a^+(x,t) \, \mathrm{d}V}{\int_{A_j(t)} |\omega|^2 |a(x,t)| \, \mathrm{d}V + \varepsilon}.$$

If many components have small $\Pi_j^+(t)$, then the fragmented support is not dominantly stretching-active. The positive-stretching reservoir decreases even if vorticity remains large.

A fragmentation alignment-loss route has the schematic form

$$\sum_j \int_{A_j(t)} |\omega|^2 a^+ \, \mathrm{d}V \quad \text{becomes lower-order}$$

because positive strain alignment is lost across the fragmented components.

This route reduces the dangerous reservoir rather than absorbing it directly by dissipation. It is useful if the remaining contribution can be bounded by lower-order enstrophy:

$$R_{\rm frag}(t) \le C_{\rm align} E_\omega(t)$$

or by an estimate with small dissipation coefficient.

4.7 Component-lifetime route

Fragments may be harmless if they are short-lived. A component that carries large instantaneous positive stretching may still contribute little to enstrophy growth if its lifetime is short relative to the local stretching time.

Let $I_j = [\tau_j, \tau_{j+1}]$ be a stopping-time interval on which component $A_j(t)$ is trackable. The integrated contribution of this component is

$$\int_{I_j} \int_{A_j(t)} |\omega|^2 a^+(x,t) \, \mathrm{d}V \, \mathrm{d}t.$$

A lifetime-control route seeks to show that the sum of such contributions is controlled:

$$\sum_j \int_{I_j} \int_{A_j(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V\ \mathrm{d}t \le \delta_{\mathrm{frag}} \int_I D(t)\ \mathrm{d}t + C_{\mathrm{frag}} \int_I E_\omega(t)\ \mathrm{d}t + C_{0,\mathrm{frag}}.$$

A useful lifetime comparison is to the local stretching time

$$\tau_{S,j}(t) \sim \frac{1}{\|S(t)\|_{L^\infty(A_j(t))} + \varepsilon},$$

or the local positive-alignment time

$$\tau_{a,j}(t) \sim \frac{1}{\|a^+(t)\|_{L^\infty(A_j(t))} + \varepsilon}.$$

If a component lifetime is much shorter than these timescales, the component may be dynamically lower-order. If many components persist across many stretching times, fragmentation remains dangerous.

4.8 Separation-cost route

Fragmentation may also produce cost through separation. When components are well separated, the strain alignment that preserves positive stretching across all components may become harder to maintain coherently. Separation may reduce nonlocal reinforcement, weaken shared alignment, or increase the cost of maintaining coordinated stretching across many pieces.

A schematic separation diagnostic is

$$d_{ij}(t) = \mathrm{dist}(A_i(t), A_j(t)).$$

One may also use a graph-based diagnostic. Let vertices represent components and weighted edges represent strain-coupling strength or positive-stretching transfer. Fragmentation becomes favorable when the graph weakens, splits, or loses coherent alignment.

A possible separation-control route is:

$$\text{large } N_{\mathrm{eff}}^+ + \text{large separation} \Longrightarrow \text{loss of coherent positive alignment or lower-order contribution.}$$

This route is especially relevant when multiple fragments remain individually coherent but collectively lose the nonlocal coordination that sustained the original coherent structure.

4.9 Fragmentation-to-scale route

Fragmentation may become scale-local. A support may look like many fragments at one scale but like a coherent structure at another, or vice versa. If the fragmented route is meaningful only after filtering, then the route exits to

$$R_{\mathrm{scale}}.$$

The channel transition is

$$R_{\mathrm{frag}} \Longrightarrow R_{\mathrm{scale}}.$$

This is not closure. It is reclassification. The contribution should be reassigned from fragmentation to scale-local control without double-counting.

A fragmentation-to-scale route is active when:

(i) component structure depends strongly on the declared observation scale;

(ii) filtering reveals organized positive stretching not visible in the raw decomposition;

(iii) fragmentation metrics are unstable under admissible scale changes;

(iv) or unresolved component structure appears below the declared scale floor.

If the declared scale family is incomplete, this is a diagnostic limitation. If the scale family is complete and the contribution remains dangerous but uncontrolled, then the route may become residual pathology.

4.10 Fragmentation-to-complement route

Fragmentation may also become threshold or complement behavior. If fragments move across the high-vorticity threshold

$$\Omega_\kappa(t) = \{x \in \Omega : |\omega(x,t)| > \kappa(t)\},$$

or if significant positive stretching lies outside this set, then the route exits to

$$R_{\text{low}}.$$

The channel transition is

$$R_{\text{frag}} \Longrightarrow R_{\text{low}}.$$

A fragmentation-to-complement route is active when:

(i) fragments flicker across the high-vorticity threshold;

(ii) the fragmented support is split between $\Omega_\kappa(t)$ and $\Omega \setminus \Omega_\kappa(t)$;

(iii) a smooth high-vorticity weight changes the channel assignment;

(iv) or the apparent fragmentation is mostly a threshold artifact.

In this case, smooth threshold splitting should be used:

$$W_\kappa(|\omega|) + \big(1 - W_\kappa(|\omega|)\big) = 1.$$

The positive stretching must be assigned to the high-vorticity side, the complement side, or a transition band. It cannot disappear through thresholding.

4.11 Fragmentation-to-pathology route

If fragmentation preserves significant positive stretching while avoiding absorbability, alignment loss, interface cost, reconnection-cost accounting, scale-local classification, complement assignment, and summable component-burst accounting, then it exits to residual pathology:

$$R_{\text{frag}} \Longrightarrow R_{\text{path}}.$$

This route is a named failure, not a hidden remainder. It says that ordinary fragmentation control has failed and that the remaining contribution belongs to the pathological concentration program.

A fragmentation-to-pathology route may involve:

(i) infinitely many components with nonsummable burst constants;

(ii) moving fragments that cannot be tracked by material, weighted, or stopping-time diagnostics;

(iii) reconnecting fragments that preserve positive stretching without paying interface or reconnection cost;

(iv) component structures that evade all declared scales;

(v) threshold-flickering fragments not controlled by R_{low};

(vi) or many components that collectively preserve positive stretching while paying too little cost.

Such a route remains an obstruction until the pathological-channel machinery reduces or absorbs it.

4.12 Conditional fragmentation control alternative

We now state the bridge target for fragmentation.

Hypothesis 4.1 (Fragmentation control alternative) *Let $R_{\text{frag}}(I)$ be active on a smooth interval*

$$I = [t_0, t_1].$$

Then at least one of the following alternatives holds:

(i) ***Pointwise absorbability:***

$$R_{\text{frag}}(t) \le \delta_{\text{frag}} D(t) + C_{\text{frag}} E_\omega(t)$$

on I.

(ii) ***Integrated absorbability:***

$$\int_{t_0}^{t} R_{\text{frag}}(s)\ \mathrm{d}s \le \delta_{\text{frag}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\text{frag}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\text{frag}}$$

for every $t \in I$.

(iii) ***Interface-cost control:*** *fragmentation creates enough magnitude-gradient, directional-gradient, boundary, or interface cost to absorb the fragmented positive stretching.*

(iv) ***Reconnection-cost control:*** *reconnection events either pay enstrophy-gradient cost, have summable integrated cost, return the route to a coherent channel, become scale-local, or exit to residual pathology.*

(v) ***Alignment loss:*** *the fragmented components lose coherent positive strain alignment, so the positive-stretching reservoir becomes lower-order.*

(vi) ***Finite component lifetime:*** *the components do not persist long enough, relative to local stretching times, to generate dangerous integrated growth.*

(vii) ***Separation weakening:*** *component separation weakens shared strain alignment or nonlocal reinforcement enough to make the fragmented contribution lower-order or absorbable.*

(viii) ***Scale-local exit:***

$$R_{\mathrm{scale}}(I)$$

activates.

(ix) ***Threshold/complement exit:***

$$R_{\mathrm{low}}(I)$$

activates.

(x) ***Residual pathological exit:***

$$R_{\mathrm{path}}(I)$$

activates.

This hypothesis is not an unconditional theorem. It is the bridge alternative needed for fragmentation to stop being an independent ordinary-channel obstruction.

4.13 Margin condition for fragmentation control

If fragmentation is controlled with coefficient δ_{frag}, then the final assembly must still satisfy

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} < 1.$$

Equivalently,

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1.$$

Therefore, fragmentation control is useful only if δ_{frag} leaves reserve for scale-local transfer, complement stretching, pathological concentration, and final coefficient recovery.

A strong fragmentation target is not merely

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} < 1.$$

The stronger useful target is

$$\delta_{\mathrm{frag}} \le (1 - \theta - \delta_{\mathrm{coh}}) - \delta_{\mathrm{reserve,frag}},$$

where

$$\delta_{\mathrm{reserve,frag}} > 0$$

is reserved for R_{scale}, R_{low}, and R_{path}.

This prevents fragmentation control from winning locally while exhausting the global proof margin.

4.14 No double-counting during fragmentation exits

When a fragmented route exits to R_{scale}, R_{low}, or R_{path}, the same positive-stretching contribution must not remain fully counted in R_{frag}. A valid channel transition requires one of the following:

(i) reassignment by stopping time;

(ii) reassignment by a partition of unity;

(iii) reassignment by smooth channel weights;

(iv) a bounded-overlap accounting rule with overlap included in the margin;

(v) or another explicit no-double-counting device.

For example, if a fragmented structure becomes scale-local after filtering, the contribution should be moved from R_{frag} to R_{scale} or split between them using weights whose overlap is charged. If fragmented components flicker across a threshold, the contribution should be split between high-vorticity and complement weights.

This rule is essential for preserving the Paper 150J [26] margin. Channel transitions are classification changes, not permission to count the same stretching several times.

4.15 What fragmentation control would prove

If the fragmentation control alternative holds and the coefficient δ_{frag} fits inside the available margin, then fragmentation does not remain an independent ordinary-channel obstruction. Every active fragmented route is either:

(i) absorbed by dissipation and lower-order enstrophy;

(ii) controlled in subinterval-stable integrated form;

(iii) made lower-order by alignment loss, finite lifetime, separation weakening, or interface cost;

(iv) paid for by reconnection cost or assigned to a coherent/scale-local/pathological exit after reconnection;

(v) reassigned to scale-local control;

(vi) reassigned to complement control;

(vii) or assigned to residual pathology.

This is progress even when fragmentation exits to another channel. It means the obstruction has been named and moved downstream rather than left inside an undefined fragmented remainder.

4.16 What this section does not prove

This section does not prove the fragmentation control alternative unconditionally. It does not prove that every fragmented support pays enough interface cost. It does not prove that every reconnection event pays enough reconnection cost. It does not prove that component separation always weakens alignment. It does not prove that all component lifetimes are short. It does not prove that all moving or reconnecting fragments have summable costs. It does not prove that every fragmented route is scale-local, complement-controlled, or pathological.

The section states the theorem target. Later analytic work must prove the required estimates with constants sharp enough to preserve the final margin.

4.17 Summary

The fragmentation channel is controlled if fragmented positive stretching is absorbed by dissipation and lower-order enstrophy, controlled in subinterval-stable integrated form, made lower-order by alignment loss or finite lifetime, paid for by interface, reconnection, or separation cost, or reassigned to R_{scale}, R_{low}, or R_{path}.

The central estimate is

$$R_{\mathrm{frag}}(t) \leq \delta_{\mathrm{frag}} D(t) + C_{\mathrm{frag}} E_{\omega}(t),$$

or its subinterval-stable integrated analogue. The estimate is useful only if the margin

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} < 1$$

remains positive.

The next section develops scale-local geometry and filtering, the second remaining ordinary channel in Paper 150N.

5 Scale-Local Geometry and Filtering

The previous sections defined fragmentation and its control targets. This section defines the second remaining ordinary channel:

$$R_{\mathrm{scale}}.$$

The scale-local channel is activated when dangerous positive stretching is not adequately visible at the full-field level, but becomes visible after filtering, coarse-graining, wave-number localization, or another declared scale decomposition.

Scale-locality is an ordinary channel, not automatically a pathology. A structure may look invisible, diffuse, filamentary, fragmented, or incoherent at one scale while becoming organized at another. The proof task is therefore to distinguish three cases:

(i) the dangerous contribution is resolved by the declared scale family and belongs to R_{scale};

(ii) the contribution is missed because the declared scale family is incomplete, which is diagnostic failure rather than theorem-level control;

(iii) the contribution evades a sufficiently complete scale family and becomes residual pathology.

The purpose of this section is to define the geometry of scale-local visibility before the control targets are stated in Section 6.

5.1 Declared scale family

Let

$$\mathcal{L} = \{\ell_1, \ell_2, \dots, \ell_N\}$$

be a declared family of spatial scales. The scales may be physical length scales, filter radii, Fourier cutoffs, dyadic shells, or other localization parameters.

For each

$$\ell \in \mathcal{L},$$

let

$$G_\ell$$

be a smoothing or localization operator at scale ℓ. In a convolution-filter formulation,

$$u_\ell = G_\ell * u, \qquad \omega_\ell = \nabla \times u_\ell,$$

and

$$S_\ell = \frac{1}{2}\left(\nabla u_\ell + \nabla u_\ell^T\right).$$

Where $|\omega_\ell| > 0$, define

$$n_\ell = \frac{\omega_\ell}{|\omega_\ell|},$$

and the scale-local strain-alignment factor

$$a_\ell(x,t) = n_{\ell,i} S_{\ell,ij} n_{\ell,j}.$$

Its positive part is

$$a_\ell^+(x,t) = \max\{a_\ell(x,t), 0\}.$$

The scale-local positive stretching density is then represented schematically by

$$|\omega_\ell|^2 a_\ell^+(x,t).$$

This quantity is not meant to replace the full Navier–Stokes stretching density. It is a diagnostic and decomposition tool for identifying where positive stretching is organized at a declared scale.

5.2 Scale-family density and resolution floor

A scale-local statement is only meaningful after the scale family has been declared. The family

$$\mathcal{L} = \{\ell_1, \ell_2, \dots, \ell_N\}$$

must be dense enough to detect the relevant scale-local support, but not so redundant that the accounting budget is inflated without control.

For theorem-level accounting, the scale family should satisfy three requirements.

First, it has a declared lower scale

$$\ell_{\min} = \min_{\ell \in \mathcal{L}} \ell.$$

In a numerical setting, $\ell_{\min}$ is constrained by grid spacing, de-aliasing, and the effective resolution of the solver. In an analytic setting, $\ell_{\min}$ should be tied to a declared viscous, functional, or localization scale.

Second, the scale family must have controlled overlap. If scale weights χ_ℓ are used, then either

$$\sum_{\ell \in \mathcal{L}} \chi_\ell(x,t) = 1$$

on the scale-local support, or

$$\sum_{\ell \in \mathcal{L}} \chi_\ell(x,t) \le K_{\text{scale}}$$

with K_{scale} explicitly charged to the margin.

Third, the scale count must not silently inflate the dissipation budget. If

$$N_{\text{scale}} = |\mathcal{L}|,$$

then the scale-local coefficient must reflect scale accounting:

$$\delta_{\text{scale,eff}} \le K_{\text{scale}} \sum_{\ell \in \mathcal{L}} \delta_\ell.$$

A dense or infinite scale family is acceptable only if the corresponding coefficient sum or integral is finite.

Thus the scale family is part of the theorem data. If $\mathcal{L}$ is too sparse, the result is diagnostic incompleteness. If $\mathcal{L}$ is too redundant and the coefficient sum diverges, the scale-local estimate fails by budget inflation.

5.3 Scale-local positive stretching

At scale ℓ, define the scale-local positive-stretching reservoir

$$P_\ell^+(t) = \int_\Omega |\omega_\ell|^2 a_\ell^+(x,t)\ \mathrm{d}V.$$

More generally, if a scale-local channel weight $\chi_\ell(x,t)$ is used, define

$$P_{\ell,\chi}^+(t) = \int_\Omega \chi_\ell(x,t) |\omega_\ell|^2 a_\ell^+(x,t)\ \mathrm{d}V.$$

A scale-local route is relevant when there exists at least one declared scale

$$\ell \in \mathcal{L}$$

for which $P_\ell^+(t)$ or $P_{\ell,\chi}^+(t)$ is non-negligible relative to the channel budget, even if the corresponding structure is not clearly visible in the full-field diagnostic.

The scale-local channel contribution is schematically

$$R_{\text{scale}}(t) \sim \sum_{\ell\in\mathcal{L}} \int_\Omega \chi_\ell(x,t)|\omega_\ell|^2 a_\ell^+(x,t)\ \mathrm{d}V,$$

with any overlap or scale-multiplicity charged through the accounting framework.

5.4 Resolved scale-local visibility

A route has resolved scale-local visibility if the declared scale family detects a non-negligible organized positive-stretching contribution.

For example, one may find a scale

$$\ell_* \in \mathcal{L}$$

such that

$$P_{\ell_*}^+(t)$$

is significant, and the corresponding support has coherent alignment, fragmented support, threshold sensitivity, or another visible structure only after filtering.

In that case, the route belongs to

$$R_{\text{scale}}.$$

This is an ordinary-channel classification. It does not yet prove control. It says only that the apparent invisibility of the route is resolved by scale localization.

The control question becomes whether the scale-local contribution pays gradient cost, cascade or transfer cost, scale-boundary cost, alignment loss, finite lifetime, complement leakage, or exits to residual pathology.

5.5 Unresolved diagnostic failure

A route should not be declared controlled or pathological merely because a coarse scale family fails to see it. If the smallest declared scale

$$\ell_{\min} = \min_{\ell\in\mathcal{L}} \ell$$

is too large to resolve the active support, then the failure is diagnostic.

This is not theorem-level scale-local control. It means the declared scale family is incomplete for the claim being made.

A typical unresolved route has the following pattern:

positive stretching is significant,

but

$$\ell_{\min}$$

is larger than the apparent support thickness, filament width, transition scale, or component separation scale.

In this case, Paper 150N should not claim that R_{scale} is controlled. The correct classification is:

$$\text{unresolved scale diagnostic.}$$

A later theorem-level statement must either refine the scale family, prove that unresolved scales pay gradient cost, or assign the route to residual pathology.

5.6 Pathological scale evasion

A stronger failure occurs when a route remains dangerous across a sufficiently complete declared scale family. This is scale evasion.

A scale-evading route satisfies, schematically:

(i) positive stretching remains significant;

(ii) no declared scale captures an absorbable or lower-order contribution;

(iii) the route avoids coherent-channel, fragmentation, and complement classifications;

(iv) unresolved high-frequency or subscale cost is not shown to be absorbable.

Such a route exits to residual pathology:

$$R_{\text{scale}} \Longrightarrow R_{\text{path}}.$$

Scale evasion should be used carefully. A route is not pathological merely because the current filters fail. It becomes pathological only after the scale family is declared sufficient for the theorem target, or after unresolved subscale activity is shown to be meaningful rather than a diagnostic artifact.

5.7 Scale family completeness

A scale-local claim requires a scale family appropriate to the problem. The declared family

$$\mathcal{L}$$

should include enough scales to distinguish resolved scale-local structure from unresolved diagnostic failure.

In a numerical setting, the lower scale is constrained by grid resolution and dealiasing. In an analytic setting, the lower scale may be tied to a physical, viscous, or functional decomposition scale. Paper 150N does not require one unique choice. It requires that the choice be declared.

A scale family is complete for a stated claim if every dangerous contribution relevant to that claim satisfies at least one of the following:

(i) it is visible at some $\ell \in \mathcal{L}$;

(ii) it pays gradient, transfer, or high-frequency cost below the declared scale;

(iii) it is lower-order;

(iv) it exits to R_{path} as a named residual obstruction.

Without such a declaration, scale-local conclusions remain diagnostic rather than theorem-level.

5.8 Filtering and positive-stretching assignment

Filtering introduces an accounting issue. The same physical positive-stretching contribution may appear at more than one scale. Therefore, scale-local channel assignment must avoid double-counting.

One approach is to use a partition over scales:

$$\sum_{\ell \in \mathcal{L}} \chi_\ell(x,t) = 1$$

on the scale-local active support. Another is to allow bounded overlap:

$$\sum_{\ell \in \mathcal{L}} \chi_\ell(x,t) \le K_{\text{scale}},$$

where $K_{\text{scale}} \ge 1$ is an overlap constant.

If $K_{\text{scale}} > 1$, then the overlap factor must be charged to the channel budget. A conservative version of the scale-local coefficient is

$$\delta_{\text{scale,eff}} = K_{\text{scale}} \delta_{\text{scale}}.$$

The final margin must then be checked using the effective coefficient.

This follows the accounting discipline of Paper 150K [27]: scale overlap is allowed only if it is paid for explicitly.

5.9 Scale-local support

A scale-local route may be supported on a filtered set

$$A_\ell(t) \subset \Omega,$$

defined by a scale-local condition such as large $|\omega_\ell|$, large a_ℓ^+, significant filtered positive stretching, or a scale-local channel weight.

A schematic filtered support is

$$A_\ell(t) = \{x \in \Omega : |\omega_\ell(x,t)| > \kappa_\ell(t)\},$$

or, more generally,

$$A_\ell(t) = \{x \in \Omega : \chi_\ell(x,t) > 0\}.$$

The scale-local positive stretching carried by $A_\ell(t)$ is

$$P^+_{A_\ell}(t) = \int_{A_\ell(t)} |\omega_\ell|^2 a_\ell^+(x,t)\ \mathrm{d}V.$$

The route belongs to R_{scale} when this filtered support carries significant positive stretching and when the corresponding full-field support is not adequately classified by coherent, fragmented, or complement geometry alone.

5.10 Scale-local fragmentation

Fragmentation may be scale-dependent. At one scale, a support may appear as a single diffuse structure. At a finer scale, it may split into many components. Conversely, many fine components may appear as one coherent or broad structure after filtering.

Thus a route may activate both fragmentation and scale-local diagnostics. The assignment must be explicit.

If the key fact is many stretching-active components at a declared scale, the route may be assigned to

$$R_{\mathrm{frag}}.$$

If the key fact is that the structure becomes visible only after filtering, the route may be assigned to

$$R_{\mathrm{scale}}.$$

If both diagnostics are essential, a bounded-overlap or partition-of-unity split should be used.

A schematic scale-fragmented route has

$$A_\ell(t) = \bigcup_j A_{\ell,j}(t),$$

with component contributions

$$P^+_{\ell,j}(t) = \int_{A_{\ell,j}(t)} |\omega_\ell|^2 a^+_\ell(x,t)\ \mathrm{d}V.$$

Such a route should be counted once, either as scale-local fragmentation inside R_{scale}, as fragmentation at scale ℓ, or by an overlap-controlled split.

5.11 Scale-local complement effects

Threshold and complement effects may also be scale-dependent. A route may lie outside the full-field high-vorticity set

$$\Omega_\kappa(t) = \{x : |\omega(x,t)| > \kappa(t)\},$$

but become visible in a filtered high-vorticity set

$$\Omega_{\kappa,\ell}(t) = \{x : |\omega_\ell(x,t)| > \kappa_\ell(t)\}.$$

Conversely, a full-field high-vorticity route may become lower-order or move into a complement region under filtering.

This produces a channel interaction:

$$R_{\mathrm{scale}} \Longleftrightarrow R_{\mathrm{low}}.$$

The assignment should be made by smooth weights whenever possible. For example, one may use a scale-dependent high-vorticity weight

$$W_{\kappa,\ell}(|\omega_\ell|)$$

and its complement

$$1 - W_{\kappa,\ell}(|\omega_\ell|).$$

The identity

$$W_{\kappa,\ell}(|\omega_\ell|) + \big(1 - W_{\kappa,\ell}(|\omega_\ell|)\big) = 1$$

ensures that filtered positive stretching is not lost through thresholding.

5.12 Scale transfer and cascade cost

A scale-local route may not stay at one scale. Positive stretching may transfer across scales, especially when structures thin, fragment, reconnect, or cascade toward smaller lengths.

A schematic scale-transfer cost measures the movement of positive stretching between scales:

$$T_{\ell\to\ell'}(t).$$

Paper 150N does not require a unique transfer definition. The theorem target is that any persistent transfer of dangerous positive stretching must satisfy one of the following:

(i) it is captured by the declared scale-local channel;

(ii) it pays gradient or high-frequency cost;

(iii) it becomes lower-order;

(iv) it exits to complement behavior;

(v) it becomes residual pathology.

The favorable route is that transfer toward smaller scales increases gradient cost, because

$$|\nabla\omega|^2$$

weights small-scale variation strongly. The dangerous route is scale transfer that preserves positive stretching while avoiding the corresponding cost.

5.13 High-frequency gradient cost

If a route becomes concentrated below a declared scale, it should generally produce high-frequency gradient cost unless it is unresolved by the diagnostic. In schematic form, smaller scales should increase contributions to

$$\int_\Omega |\nabla\omega|^2 \, \mathrm{d}V.$$

A scale-local cost target may therefore take the form

$$R_{\mathrm{scale}}(t) \le c_{\mathrm{hf}}\nu \int_\Omega |\nabla\omega|^2 \, \mathrm{d}V + C_{\mathrm{hf}} E_\omega(t),$$

or more specifically,

$$R_{\mathrm{scale}}(t) \le c_{\mathrm{hf}} D_{\mathrm{hf}}(t) + C_{\mathrm{hf}} E_\omega(t),$$

where $D_{\mathrm{hf}}(t)$ denotes the part of dissipation associated with the active high-frequency or unresolved scale band.

The difficult case is:

$$R_{\mathrm{scale}}(t) \text{ large,} \qquad D_{\mathrm{hf}}(t) \text{ too small.}$$

Such a route would mean that scale-local positive stretching survives without paying the gradient cost expected from small-scale localization.

5.14 Scale-local lifetime

Scale-local structures may be short-lived. A filtered feature at scale ℓ may appear briefly, transfer to another scale, or dissolve. The relevant quantity is therefore often integrated positive stretching:

$$\int_{I_\ell} P_\ell^+(t)\ \mathrm{d}t,$$

where I_ℓ is a time interval on which the scale-local structure is active.

A scale-local route is dangerous if its cumulative contribution is significant:

$$\sum_{\ell \in \mathcal{L}} \int_{I_\ell} P_\ell^+(t)\ \mathrm{d}t$$

and not absorbable by dissipation, enstrophy, or controlled residual constants.

If scale-local events occur as bursts, then the burst constants must be summable:

$$\sum_m C_{0,m}^{\mathrm{scale}} < \infty.$$

Otherwise, many individually controlled scale-local events could accumulate into an uncontrolled contribution.

5.15 Definition of scale-local channel

We now give the working definition.

Definition 5.1 (Scale-local channel) *Let u be a smooth solution of the three-dimensional incompressible Navier–Stokes equations on*

$$I = [t_0, t_1].$$

Let

$$\mathcal{L} = \{\ell_1, \ell_2, \ldots, \ell_N\}$$

be a declared scale family with associated filters G_ℓ, scale-local fields $u_\ell, \omega_\ell, S_\ell$, and scale-local alignment factors a_ℓ^+.

The scale-local channel $R_{\mathrm{scale}}(I)$ is active on I if there exists at least one scale $\ell \in \mathcal{L}$, or a family of active scales, such that:

(i) the scale-local positive-stretching reservoir

$$\int_I \int_\Omega \chi_\ell(x,t)|\omega_\ell|^2 a_\ell^+(x,t)\ \mathrm{d}V\ \mathrm{d}t$$

is non-negligible relative to the channel budget;

(ii) the dangerous contribution is not adequately visible or classifiable at the full-field level alone;

(iii) the scale-local support is resolved by the declared scale family, or the failure of resolution is explicitly recorded as diagnostic incompleteness;

(iv) overlap across scales is controlled by a partition, bounded-overlap rule, or equivalent accounting device;

(v) the scale budget is finite, meaning that the scale-overlap coefficient and the sum or integral of scale-local coefficients are controlled;

(vi) the contribution is not already assigned entirely to primary entry, coherent aligned-patch support, transition-layer support, fragmentation, low-vorticity complement stretching, or residual pathology.

This definition makes R_{scale} a visibility channel for filtered or scale-dependent positive stretching. It does not prove scale-local control.

5.16 Scale-local exits

Once scale-local visibility is active, several outcomes are possible.

First, the scale-local contribution may be directly absorbable:

$$R_{\mathrm{scale}}(t) \le \delta_{\mathrm{scale}} D(t) + C_{\mathrm{scale}} E_\omega(t).$$

Second, it may be controlled in subinterval-stable integrated form:

$$\int_{t_0}^{t} R_{\mathrm{scale}}(s)\ \mathrm{d}s \le \delta_{\mathrm{scale}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\mathrm{scale}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\mathrm{scale}}$$

for every $t \in I$.

Third, it may pay high-frequency, transfer, or gradient cost.

Fourth, it may become lower-order through alignment loss, finite lifetime, or decay of scale-local positive stretching.

Fifth, it may exit to the low-vorticity complement channel:

$$R_{\mathrm{scale}} \Longrightarrow R_{\mathrm{low}},$$

if the scale-local support is threshold-dependent.

Sixth, it may exit to residual pathology:

$$R_{\mathrm{scale}} \Longrightarrow R_{\mathrm{path}},$$

if the contribution remains dangerous while evading the declared scale family, finite scale-budget accounting, and all ordinary control routes.

5.17 What scale-local geometry does not prove

This section defines scale-local geometry and filtering. It does not prove scale-local control.

In particular, it does not prove that every scale-local route pays enough gradient cost. It does not prove that every declared scale family is complete. It does not prove that every scale family has finite budget automatically. It does not prove that scale transfer is absorbable. It does not prove that high-frequency concentration always becomes dissipative. It does not prove that scale-local bursts have summable constants. It does not prove that unresolved scale-local structures are pathological rather than diagnostic failures.

Those are control questions. They are stated as theorem targets in Section 6.

5.18 Summary

The scale-local channel R_{scale} is activated when dangerous positive stretching becomes visible only after filtering or scale decomposition. The channel requires a declared scale family, an accounting rule for overlap across scales, a finite scale budget, and a distinction between resolved scale-local visibility, unresolved diagnostic failure, and residual scale evasion.

A scale-local route may become absorbable, pay gradient or transfer cost, become lower-order, enter the complement channel, or become residual pathology. The next section formulates the control targets for R_{scale}.

6 Scale-Local Control Targets

The previous section defined scale-local geometry and filtering. This section formulates the control targets for the scale-local channel

$$R_{\text{scale}}.$$

The goal is not to prove unconditional scale-local control in this paper. The goal is to state what must be shown for filtered or scale-dependent positive stretching to become absorbable, lower-order, or a named residual obstruction in the Paper 150J [26] assembly.

The central question is:

> If dangerous stretching is visible only after filtering, does scale localization force cost?

The favorable answer is that scale-local visibility either produces gradient cost, transfer cost, finite lifetime, alignment loss, complement behavior, or a lower-order contribution. The dangerous answer is that positive stretching survives across scales while avoiding both full-field control and scale-local cost.

6.1 The scale-local obstruction

A scale-local route is obstructive when a declared scale

$$\ell \in \mathcal{L}$$

or a family of scales carries significant positive stretching:

$$P^{+}_{\ell,\chi}(t) = \int_{\Omega} \chi_\ell(x,t)|\omega_\ell|^2 a^{+}_\ell(x,t)\ \mathrm{d}V,$$

while the corresponding scale-local cost remains too small to absorb it.

The strongest obstruction has the following features:

(i) the full-field diagnostic does not adequately classify the route;

(ii) at least one declared scale reveals significant positive stretching;

(iii) the scale-local support does not reduce to coherent aligned-patch support, transition-layer support, fragmentation, or complement behavior;

(iv) high-frequency, gradient, or transfer cost remains too small;

(v) the route persists long enough to contribute to integrated growth;

(vi) the route does not become residual pathology in a controlled way.

In schematic form, the dangerous regime is

$$R_{\mathrm{scale}}(t) \text{ large}, \qquad D_{\mathrm{scale}}(t) \text{ too small},$$

where $D_{\mathrm{scale}}(t)$ denotes the dissipation, gradient, transfer, or high-frequency cost associated with the active scale-local structure.

6.2 Pointwise scale-local target

The strongest useful target is a pointwise estimate:

$$R_{\mathrm{scale}}(t) \le \delta_{\mathrm{scale}} D(t) + C_{\mathrm{scale}} E_\omega(t).$$

Here $\delta_{\mathrm{scale}} \ge 0$ is the dissipation fraction consumed by the scale-local channel, and $C_{\mathrm{scale}} \ge 0$ is a lower-order enstrophy coefficient.

This estimate is useful for the Paper 150J assembly only if the final margin remains positive:

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} < 1.$$

Equivalently,

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1, \qquad \delta_{\mathrm{rem,ord}} = \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}}.$$

Thus scale-local control must not merely identify filtered structure. It must control that structure with a coefficient sharp enough to preserve reserve for complement behavior, residual pathology, and final coefficient recovery.

6.3 Integrated scale-local target

Scale-local structures may be moving, intermittent, or active only on certain scale-time windows. A pointwise estimate may be too strong in such cases. The appropriate target is then a subinterval-stable integrated estimate:

$$\int_{t_0}^{t} R_{\mathrm{scale}}(s)\ \mathrm{d}s \leq \delta_{\mathrm{scale}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\mathrm{scale}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\mathrm{scale}}$$

for every

$$t \in I = [t_0, t_1].$$

This estimate must hold on every subinterval or through stopping-time partial sums that control every intermediate time. Otherwise, a full-interval bound may hide a transient scale-local spike.

If scale-local activity is decomposed into burst intervals

$$I_m = [\tau_m, \tau_{m+1}],$$

then the residual constants must satisfy

$$\sum_m C_{0,m}^{\mathrm{scale}} < \infty,$$

or be controlled by a finite budget. Without this summability, infinitely many short scale-local events could accumulate into an uncontrolled contribution.

6.4 High-frequency gradient-cost route

The most direct scale-local control mechanism is high-frequency gradient cost. If positive stretching becomes concentrated at smaller scales, then one expects the gradient term

$$\int_\Omega |\nabla \omega|^2\ \mathrm{d}V$$

to increase.

Let $D_{\mathrm{hf}}(t)$ denote the dissipation associated with the active high-frequency or small-scale portion of the flow. A schematic high-frequency cost estimate is

$$R_{\mathrm{scale}}(t) \leq c_{\mathrm{hf}} D_{\mathrm{hf}}(t) + C_{\mathrm{hf}} E_\omega(t).$$

Since

$$D_{\mathrm{hf}}(t) \leq D(t),$$

this becomes useful for the final assembly if the effective coefficient fits into the margin:

$$c_{\mathrm{hf}} \leq \delta_{\mathrm{scale}}$$

with

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} < 1.$$

The dangerous case is scale-local positive stretching without high-frequency cost:

$$R_{\mathrm{scale}}(t) \text{ large}, \qquad D_{\mathrm{hf}}(t) \text{ small}.$$

Such a route would mean that filtering reveals organized positive stretching but the expected small-scale gradient penalty does not appear.

6.5 Scale-transfer route

A scale-local contribution may move between scales. Positive stretching may be transferred from a broad support into smaller structures, from small structures back into larger coherent forms, or across several active bands.

Let

$$T_{\ell\to\ell'}(t)$$

denote a schematic transfer of positive-stretching activity from scale ℓ to scale ℓ'. Paper 150N does not require a unique transfer definition. The control target is that persistent scale transfer must be accounted for by one of the following:

(i) transfer produces gradient or high-frequency cost;

(ii) transfer weakens positive alignment;

(iii) transfer makes the route lower-order;

(iv) transfer exits to fragmentation or complement behavior;

(v) transfer becomes residual pathology.

A scale-transfer estimate may take the integrated form

$$\sum_{\ell,\ell'} \int_{t_0}^{t} T_{\ell\to\ell'}(s)\ \mathrm{d}s \le \delta_{\mathrm{scale}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\mathrm{scale}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\mathrm{transferscale}}.$$

The estimate is useful only if it is subinterval-stable and if the transfer constants are summable across repeated scale transitions.

6.6 Scale-boundary cost route

A filtered support may have boundaries or transition regions at scale ℓ. Let

$$A_\ell(t)$$

be a scale-local support. A schematic scale-boundary cost is

$$B_\ell(t) = \nu \int_{N_\ell(\partial A_\ell(t))} |\nabla\omega|^2\ \mathrm{d}V,$$

where $N_\ell(\partial A_\ell(t))$ is a neighborhood of the filtered support boundary with thickness comparable to ℓ.

If the scale-local support preserves positive stretching while remaining sharply separated from its surroundings, then it should pay boundary or transition cost. A possible control route is

$$R_{\mathrm{scale}}(t) \le c_{\mathrm{bd,scale}} \sum_{\ell\in\mathcal{L}} B_\ell(t) + C_{\mathrm{bd,scale}} E_\omega(t).$$

The difficult case is scale-local support that remains organized and positively stretching while paying too little boundary cost.

6.7 Alignment-loss route

Scale localization may reduce positive strain alignment. At scale ℓ, the relevant alignment factor is

$$a_\ell(x,t) = n_{\ell,i} S_{\ell,ij} n_{\ell,j},$$

with positive part

$$a_\ell^+(x,t) = \max\{a_\ell(x,t), 0\}.$$

A scale-local aligned fraction may be defined by

$$\Pi_\ell^+(t) = \frac{\int_\Omega \chi_\ell |\omega_\ell|^2 a_\ell^+ \, \mathrm{d}V}{\int_\Omega \chi_\ell |\omega_\ell|^2 |a_\ell| \, \mathrm{d}V + \varepsilon}.$$

If $\Pi_\ell^+(t)$ decreases, then the scale-local structure no longer carries dominantly positive stretching.

This route makes R_{scale} lower-order by reducing the positive-stretching reservoir itself. A schematic alignment-loss control statement is

$$R_{\mathrm{scale}}(t) \le C_{\mathrm{align,scale}} E_\omega(t)$$

after the positive alignment falls below a declared threshold.

6.8 Finite scale-local lifetime

A scale-local structure may be visible but short-lived. It may appear only during a transient thinning, burst, reconnection, or filtered alignment event. Such a route is dangerous only if it lasts long enough to accumulate significant positive stretching.

Let

$$I_{\ell,m} = [\tau_{\ell,m}, \tau_{\ell,m+1}]$$

be a stopping-time interval on which scale ℓ is active. The integrated scale-local contribution is

$$\int_{I_{\ell,m}} P_{\ell,\chi}^+(t) \, \mathrm{d}t.$$

A finite-lifetime route seeks to prove that

$$\sum_{\ell,m} \int_{I_{\ell,m}} P_{\ell,\chi}^+(t) \, \mathrm{d}t$$

is controlled by dissipation, enstrophy, and summable constants.

If a scale-local event lasts much less than the local stretching time,

$$\tau_{\ell,S}(t) \sim \frac{1}{\|S_\ell(t)\|_{L^\infty(A_\ell(t))} + \varepsilon},$$

then it may be lower-order. If it persists across many local stretching times, it becomes a genuine scale-local obstruction unless it pays cost or exits to another channel.

6.9 Resolved-scale absorbability

A scale-local route is resolved if a declared scale $\ell \in \mathcal{L}$ captures the positive-stretching contribution. Once resolved, the target is absorbability:

$$R_{\text{scale}}(t) \leq \delta_{\text{scale}} D(t) + C_{\text{scale}} E_\omega(t).$$

A stronger resolved-scale statement may be written at each active scale:

$$R_\ell(t) \leq \delta_\ell D(t) + C_\ell E_\omega(t),$$

with

$$\sum_{\ell \in \mathcal{L}} \delta_\ell \leq \delta_{\text{scale}}$$

after overlap accounting.

If the scale weights have bounded overlap

$$\sum_{\ell \in \mathcal{L}} \chi_\ell(x,t) \leq K_{\text{scale}},$$

then the effective coefficient becomes

$$\delta_{\text{scale,eff}} = K_{\text{scale}} \sum_{\ell \in \mathcal{L}} \delta_\ell.$$

The final margin must be checked using $\delta_{\text{scale,eff}}$.

6.10 Finite and infinite scale budgets

The scale-local channel may be formulated with a finite scale family or with an idealized infinite family. The accounting requirement is different in the two cases.

For a finite scale family

$$\mathcal{L} = \{\ell_1, \ldots, \ell_N\},$$

one may use the coefficient sum

$$\delta_{\text{scale}} = \sum_{\ell \in \mathcal{L}} \delta_\ell$$

after overlap adjustment.

For an infinite or continuum scale family, the coefficient budget must be summable or integrable. Schematically, one needs

$$\sum_{\ell \in \mathcal{L}} \delta_\ell < \infty$$

in a countable formulation, or

$$\int_{\mathcal{L}} \delta(\ell) \, \mathrm{d}\mu(\ell) < \infty$$

in a continuum formulation.

Without such a condition, scale-local control may fail even if every individual scale is locally estimated, because the total scale budget can diverge. This is the scale-local analogue of nonsummable burst constants.

Therefore, Paper 150N treats scale-family density as part of the bridge hypothesis:

$$\text{scale-local control} = \text{resolved scales} + \text{controlled overlap} + \text{finite scale budget}.$$

6.11 Unresolved-scale cost

If the dangerous route lies below the declared scale floor $\ell_{\min}$, then a theorem-level result must not simply declare success. One must show that the unresolved contribution pays cost, becomes lower-order, or is assigned to residual pathology.

A schematic unresolved-scale alternative is:

$$\text{unresolved positive stretching} \implies \text{gradient cost, lower-order behavior, or } R_{\text{path}}.$$

The gradient-cost version may be expressed as

$$R_{<\ell_{\min}}(t) \le \delta_{<\ell_{\min}} D(t) + C_{<\ell_{\min}} E_\omega(t),$$

where $R_{<\ell_{\min}}$ denotes the unresolved subscale contribution.

If no such estimate is available, then the correct conclusion is not control. It is diagnostic incompleteness or residual scale pathology.

6.12 Scale-local to complement exit

A scale-local route may depend on thresholds. A filtered structure may appear inside a scale-local high-vorticity set

$$\Omega_{\kappa,\ell}(t) = \{x : |\omega_\ell(x,t)| > \kappa_\ell(t)\}$$

but outside the full-field high-vorticity set, or vice versa.

If threshold dependence is essential, the route exits to the complement channel:

$$R_{\text{scale}} \implies R_{\text{low}}.$$

This exit is active when:

(i) scale-local positive stretching lies outside the selected high-vorticity mask;

(ii) scale-local support flickers across a threshold;

(iii) smooth high-vorticity weights alter the classification;

(iv) complement splitting is needed to keep positive stretching from disappearing.

A valid scale-to-complement exit must reassign the contribution without double-counting.

6.13 Scale-local to pathology exit

If a scale-local route remains dangerous while avoiding resolved-scale absorbability, high-frequency cost, transfer cost, alignment loss, finite lifetime, complement assignment, finite scale-budget control, and lower-order control, then it exits to residual pathology:

$$R_{\text{scale}} \Longrightarrow R_{\text{path}}.$$

This is the named scale-evasion failure. It says that the route remains dangerous after the declared scale family has done all the work it can do.

A scale-local-to-pathology route may involve:

(i) positive stretching concentrated below the declared scale floor;

(ii) scale transfer with nonsummable constants;

(iii) filtered support that remains visible but not absorbable;

(iv) high-frequency activity without sufficient gradient cost;

(v) scale-sensitive support that cannot be classified robustly;

(vi) scale-local bursts that are not summable;

(vii) a scale family whose coefficient sum diverges.

Such a route remains an obstruction until the pathological-channel layer reduces or absorbs it.

6.14 Conditional scale-local control alternative

We now state the bridge target for scale-local control.

Hypothesis 6.1 (Scale-local control alternative) *Let $R_{\text{scale}}(I)$ be active on a smooth interval*

$$I = [t_0, t_1].$$

Then at least one of the following alternatives holds:

(i) ***Pointwise absorbability:***

$$R_{\text{scale}}(t) \le \delta_{\text{scale}} D(t) + C_{\text{scale}} E_\omega(t)$$

on I.

(ii) ***Integrated absorbability:***

$$\int_{t_0}^{t} R_{\text{scale}}(s)\ \mathrm{d}s \le \delta_{\text{scale}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\text{scale}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\text{scale}}$$

for every $t \in I$.

(iii) ***High-frequency cost:*** *the scale-local contribution pays enough gradient or high-frequency dissipation to become absorbable.*

(iv) ***Scale-transfer cost:*** *transfer across scales is controlled by dissipation, enstrophy, and summable residual constants.*

(v) ***Scale-boundary cost:*** *filtered supports pay enough boundary or transition cost.*

(vi) ***Alignment loss:*** *scale-local positive strain alignment weakens enough to make the contribution lower-order.*

(vii) ***Finite scale-local lifetime:*** *scale-local structures do not persist long enough, relative to local stretching times, to generate dangerous integrated growth.*

(viii) ***Resolved-scale accounting:*** *the declared scale family captures the contribution with controlled overlap and margin-compatible coefficients.*

(ix) ***Finite scale-budget control:*** *the active scale family has finite overlap-adjusted coefficient budget, whether the scale family is finite, countable, or continuum-indexed.*

(x) ***Complement exit:***

$$R_{\mathrm{low}}(I)$$

activates.

(xi) ***Residual pathological exit:***

$$R_{\mathrm{path}}(I)$$

activates.

This hypothesis is not an unconditional theorem. It is the bridge alternative needed for scale-local visibility to stop being an independent ordinary-channel obstruction.

6.15 Margin condition for scale-local control

If scale-local control consumes coefficient δ_{scale}, then the final assembly must satisfy

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} < 1.$$

Equivalently,

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1.$$

Thus scale-local estimates are useful only if

$$\delta_{\mathrm{scale}}$$

does not consume the remaining dissipation reserve.

A stronger practical target is

$$\delta_{\mathrm{scale}} \le (1 - \theta - \delta_{\mathrm{coh}} - \delta_{\mathrm{frag}}) - \delta_{\mathrm{reserve,scale}},$$

where

$$\delta_{\mathrm{reserve,scale}} > 0$$

is reserved for complement control and residual pathology.

This prevents scale-local control from succeeding locally while exhausting the budget needed by downstream channels.

6.16 No double-counting across scales and exits

Scale-local control has a high risk of double-counting because the same positive-stretching contribution may appear at several scales, and may also be fragmented or threshold-dependent.

A valid scale-local accounting scheme must specify one of the following:

(i) a partition over active scales;

(ii) a bounded-overlap scale family with overlap charged to the margin;

(iii) a stopping-time scale assignment;

(iv) a finite or summable scale-budget rule;

(v) a scale-by-channel partition separating R_{frag}, R_{scale}, and R_{low};

(vi) or an equivalent bookkeeping device.

If a scale-local route exits to R_{low}, it should not remain fully counted in R_{scale}. If a scale-local route is also fragmented, the contribution should be split or assigned by a declared rule. If residual scale evasion exits to R_{path}, the independent R_{scale} contribution should be removed or overlap-charged.

This bookkeeping is necessary for the margin condition to be meaningful.

6.17 What scale-local control would prove

If the scale-local control alternative holds and the coefficient δ_{scale} fits inside the available margin, then scale-local visibility does not remain an independent ordinary-channel obstruction. Every active scale-local route is either:

(i) absorbed pointwise;

(ii) absorbed in subinterval-stable integrated form;

(iii) controlled by high-frequency, scale-transfer, or scale-boundary cost;

(iv) made lower-order by alignment loss or finite lifetime;

(v) captured by resolved-scale accounting;

(vi) controlled by a finite scale budget;

(vii) reassigned to complement control;

(viii) or assigned to residual pathology.

This is progress even when the route exits to another channel. It means scale-local invisibility has been converted into a named dependency.

6.18 What this section does not prove

This section does not prove the scale-local control alternative unconditionally. It does not prove that every scale-local route pays enough high-frequency cost. It does not prove that every scale family is complete. It does not prove that every scale family has finite coefficient budget automatically. It does not prove that scale transfer is always summable. It does not prove that unresolved activity is lower-order. It does not prove that every filtered structure is absorbable. It does not prove that all scale-local threshold effects are controlled by R_{low}.

The section states the theorem target. Later analytic work must prove the estimates with constants sharp enough to preserve the final margin.

6.19 Summary

The scale-local channel is controlled if filtered positive stretching is absorbed by dissipation and lower-order enstrophy, controlled in subinterval-stable integrated form, paid for by high-frequency, transfer, or scale-boundary cost, made lower-order by alignment loss or finite lifetime, captured by finite scale-budget accounting, or reassigned to R_{low} or R_{path}.

The central estimate is

$$R_{\mathrm{scale}}(t) \le \delta_{\mathrm{scale}} D(t) + C_{\mathrm{scale}} E_\omega(t),$$

or its subinterval-stable integrated analogue. The estimate is useful only if

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} < 1$$

remains positive.

The next section develops the low-vorticity complement and threshold-flicker geometry needed to control R_{low}.

7 Low-Vorticity Complement and Threshold-Flicker Geometry

The previous sections defined fragmentation and scale-local visibility. This section defines the third remaining ordinary channel studied in Paper 150N:

$$R_{\mathrm{low}}.$$

The low-vorticity complement channel is activated when positive stretching lies outside the selected high-vorticity region, near a threshold boundary, or inside the complement of a smooth high-vorticity weight.

This channel is necessary because high-vorticity arguments depend on a choice of threshold. A dangerous contribution should not disappear merely because a cutoff missed it. If positive stretching slips below a high-vorticity mask, flickers across a threshold, or lies in a smooth transition band, then it must be counted. The complement channel is the ordinary-channel mechanism that prevents thresholding from becoming a loophole.

7.1 High-vorticity masks and complements

Let

$$\kappa(t) > 0$$

be a high-vorticity threshold. The sharp high-vorticity region is

$$\Omega_\kappa(t) = \{x \in \Omega : |\omega(x,t)| > \kappa(t)\}.$$

The complement is

$$\Omega \setminus \Omega_\kappa(t) = \{x \in \Omega : |\omega(x,t)| \le \kappa(t)\}.$$

The positive stretching inside the high-vorticity region is

$$P_\kappa^+(t) = \int_{\Omega_\kappa(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V.$$

The positive stretching in the complement is

$$P_{\mathrm{low}}^+(t) = \int_{\Omega \setminus \Omega_\kappa(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V.$$

The total positive stretching splits exactly:

$$P^+(t) = P_\kappa^+(t) + P_{\mathrm{low}}^+(t).$$

Thus the high-vorticity mask does not remove positive stretching. It only assigns positive stretching to a high-vorticity part and a complement part. If the complement part is significant, then R_{low} is active.

7.2 Smooth high-vorticity weights

Sharp thresholds can create artificial discontinuities. A smooth high-vorticity weight gives a more robust decomposition. Let

$$W_\kappa : [0,\infty) \to [0,1]$$

be a smooth nondecreasing weight such that $W_\kappa(r)$ is small for $r \ll \kappa$ and close to one for $r \gg \kappa$. The complement weight is

$$1 - W_\kappa(|\omega|).$$

The identity

$$W_\kappa(|\omega|) + \big(1 - W_\kappa(|\omega|)\big) = 1$$

gives the exact splitting

$$\int_\Omega |\omega|^2 a^+ \, \mathrm{d}V = \int_\Omega W_\kappa(|\omega|)|\omega|^2 a^+ \, \mathrm{d}V + \int_\Omega (1 - W_\kappa(|\omega|))|\omega|^2 a^+ \, \mathrm{d}V.$$

The first term is the weighted high-vorticity contribution. The second term is the weighted complement contribution. A smooth-weight formulation is often preferable because it avoids treating threshold-crossing as an artificial jump.

7.3 Threshold bands

A smooth threshold has a transition band. For parameters

$$0 < \kappa_- < \kappa_+,$$

define the threshold band

$$B_\kappa(t) = \{x \in \Omega : \kappa_-(t) \le |\omega(x,t)| \le \kappa_+(t)\}.$$

Positive stretching in this band is

$$P^+_{B_\kappa}(t) = \int_{B_\kappa(t)} |\omega|^2 a^+(x,t) \, \mathrm{d}V.$$

The threshold band is important because dangerous stretching may sit neither clearly in the high-vorticity region nor clearly in a harmless low-vorticity region. If positive stretching concentrates near the threshold boundary, then a sharp cutoff may misclassify it. The complement channel must include this transition-band contribution unless it is controlled elsewhere.

A threshold-band route is favorable if the band contribution is lower-order, controlled by smooth-weight estimates, or absorbed by dissipation and enstrophy. It is dangerous if the band repeatedly carries significant positive stretching while avoiding both high-vorticity control and complement control.

7.4 Low-vorticity complement stretching

The phrase "low-vorticity complement" does not mean zero vorticity. It means the region outside the selected high-vorticity mask or outside the high-weight part of a smooth threshold. The complement may still contain nontrivial vorticity and positive stretching.

The complement contribution is schematically

$$R_{\mathrm{low}}(t) \sim \int_{\Omega \setminus \Omega_\kappa(t)} |\omega|^2 a^+(x,t) \, \mathrm{d}V,$$

or, in smooth-weight form,

$$R_{\mathrm{low}}(t) \sim \int_\Omega (1 - W_\kappa(|\omega|))|\omega|^2 a^+(x,t) \, \mathrm{d}V.$$

A complement route is dangerous if this contribution remains significant relative to the available budget. It is favorable if the lower vorticity magnitude allows control by enstrophy, if positive alignment weakens, if the contribution is absorbed by dissipation, or if the route is reassigned to a threshold-flicker or residual pathological channel.

7.5 Why complement control is necessary

The high-vorticity pinching program focuses on dangerous amplification in high-vorticity regimes. However, a proof cannot simply ignore stretching outside a chosen high-vorticity set.

There are several reasons:

(i) positive stretching may lie just below the threshold;

(ii) a route may flicker across the threshold repeatedly;

(iii) broad low-intensity alignment may contribute significantly over large measure;

(iv) a filtered or fragmented route may move between high-vorticity and complement regions;

(v) the selected threshold may not match the dynamically active support.

The complement channel prevents these cases from leaking out of the accounting system. If a contribution is not controlled in the high-vorticity region, then it must appear in the complement, a transition band, a scale-local channel, or residual pathology.

7.6 Threshold flicker

Threshold flicker occurs when a stretching-active support repeatedly crosses the high-vorticity boundary. For a sharp threshold, this means that a support moves in and out of

$$\Omega_\kappa(t) = \{x : |\omega(x,t)| > \kappa(t)\}.$$

For a smooth threshold, it means that the support moves through the transition band where

$$0 < W_\kappa(|\omega|) < 1.$$

Let

$$I_m = [\tau_m, \tau_{m+1}]$$

be stopping-time intervals associated with threshold crossings or bursts of complement stretching. The cumulative flicker contribution is

$$\sum_m \int_{I_m} \int_\Omega (1 - W_\kappa(|\omega|))|\omega|^2 a^+ \, \mathrm{d}V \, \mathrm{d}t.$$

Threshold flicker is dangerous if this cumulative contribution is significant and not absorbable by dissipation, enstrophy, or controlled residual constants. It is not harmless merely because each individual crossing is short.

7.7 Zeno-style threshold flicker

A sharper form of threshold flicker is Zeno-style flicker: infinitely many threshold crossings occur in a finite time interval. Let

$$I_m = [\tau_m, \tau_{m+1}], \qquad \tau_m \to T < \infty.$$

Even if each flicker interval appears individually controlled, the total contribution may fail if the residual constants are not summable:

$$\sum_m C_{0,m}^{\mathrm{low}} = \infty.$$

A valid complement-channel estimate must prevent this. Either the flicker intervals consume a finite budget, their constants are summable, or the route must be assigned to residual pathology:

$$R_{\mathrm{low}} \Longrightarrow R_{\mathrm{path}}.$$

This is the complement-channel analogue of the burst-summability requirement from Paper 150K [27].

7.8 Finite-crossing or summable-flicker requirement

Threshold flicker is controlled only if repeated threshold crossings do not generate an uncontrolled cumulative contribution. A valid complement-control argument must establish at least one of the following:

(i) only finitely many significant threshold crossings occur on the interval;

(ii) infinitely many crossings occur, but their residual constants are summable;

(iii) each significant crossing consumes a fixed positive amount of a finite budget, such as dissipation, transition-band cost, alignment-loss cost, or threshold-motion cost;

(iv) the cumulative positive stretching during the flicker sequence is directly absorbable;

(v) the route is assigned to R_{path} as residual threshold pathology.

In formulas, for threshold-flicker intervals I_m, one needs either a finite number of significant intervals or

$$\sum_m C_{0,m}^{\mathrm{low}} < \infty.$$

A stronger finite-budget mechanism is

$$C_{0,m}^{\mathrm{low}} \le c_0 \int_{I_m} B_{\mathrm{flicker}}(t)\ \mathrm{d}t + b_m, \qquad \sum_m b_m < \infty,$$

with

$$\int_I B_{\mathrm{flicker}}(t)\ \mathrm{d}t < \infty.$$

Then

$$\sum_m C_{0,m}^{\mathrm{low}} < \infty.$$

Thus infinitely fast threshold flicker is not allowed to remain inside R_{low} for free. It must be finite, summable, budget-paid, absorbable, or pathological.

7.9 Broad low-intensity complement support

Complement stretching may be broad rather than localized. A region with modest vorticity and weak positive alignment can still contribute significantly if it has large measure:

$$\int_{\Omega \setminus \Omega_\kappa(t)} |\omega|^2 a^+ \, \mathrm{d}V$$

may be non-negligible even when $|\omega| \leq \kappa(t)$.

This is a low-vorticity analogue of wide-area low-intensity danger. It is favorable if the contribution is bounded by enstrophy:

$$|\omega|^2 \leq \kappa(t)^2 \quad \text{on } \Omega \setminus \Omega_\kappa(t),$$

together with a bound on the positive alignment factor. It is dangerous if weak positive stretching over broad support accumulates enough to affect the enstrophy balance.

Thus complement control must include both localized threshold effects and broad low-intensity support.

7.10 Complement and lower-order enstrophy control

The complement region has the useful property

$$|\omega| \leq \kappa(t)$$

for a sharp threshold. Therefore,

$$|\omega|^2 a^+ \leq \kappa(t)^2 a^+$$

on $\Omega \setminus \Omega_\kappa(t)$. This alone does not control complement stretching, because a^+ depends on the strain field. But it suggests that complement stretching may be lower-order if the positive alignment factor can be bounded in a suitable norm.

A schematic lower-order route is

$$R_{\mathrm{low}}(t) \leq C_{\mathrm{low}} E_\omega(t),$$

or

$$R_{\mathrm{low}}(t) \leq \delta_{\mathrm{low}} D(t) + C_{\mathrm{low}} E_\omega(t)$$

with small δ_{low}.

The difficult case is complement stretching that is not high-vorticity but still carries enough positive strain alignment to affect the enstrophy balance.

7.11 Complement and scale-locality

Complement behavior may be scale-local. A route may lie outside the full-field high-vorticity mask while becoming visible at a filtered scale:

$$|\omega(x,t)| \leq \kappa(t), \qquad |\omega_\ell(x,t)| > \kappa_\ell(t).$$

Alternatively, a full-field high-vorticity route may become complement-like after filtering.

This interaction activates a channel transition:

$$R_{\text{low}} \Longleftrightarrow R_{\text{scale}}.$$

The assignment must be explicit. If the essential feature is threshold leakage, the contribution belongs to R_{low}. If the essential feature is filtered visibility, it belongs to R_{scale}. If both are essential, a partition or bounded-overlap accounting rule must be used.

7.12 Complement and fragmentation

Complement support may be fragmented. Many components may lie outside the high-vorticity mask while collectively carrying positive stretching:

$$\Omega \setminus \Omega_\kappa(t) \supset \bigcup_j A_j(t).$$

This activates an interaction between

$$R_{\text{frag}} \quad \text{and} \quad R_{\text{low}}.$$

If the main issue is many components, the route may be assigned to fragmentation. If the main issue is that the components lie outside the high-vorticity mask or flicker across the threshold, the route may be assigned to complement control. If both are essential, the contribution should be split by weights or overlap-charged.

The no-double-counting rule remains essential. A fragmented complement contribution should not be fully charged to both R_{frag} and R_{low} unless the overlap is included in the margin.

7.13 Smooth-threshold robustness

A threshold-based conclusion should be robust under reasonable changes of the threshold or smooth weight. If a contribution is classified as controlled for one cutoff but dangerous for a nearby cutoff, then the result is threshold-sensitive.

Let

$$W_\kappa^{(\rho)}(|\omega|)$$

be a family of smooth high-vorticity weights indexed by a transition-band thickness or smoothing parameter ρ. The complement contribution is

$$R_{\text{low}}^{(\rho)}(t) = \int_\Omega (1 - W_\kappa^{(\rho)}(|\omega|))|\omega|^2 a^+ \, \mathrm{d}V.$$

A robust complement claim should remain qualitatively stable for admissible variations of ρ. If the conclusion changes under small smooth changes of the threshold band, then the claim is diagnostic rather than theorem-level.

Thus complement control should preferably be stated using smooth weights, threshold bands, or stability under nearby cutoffs.

7.14 Threshold motion

The threshold $\kappa(t)$ may be time-dependent. For example, it may be chosen as a quantile of $|\omega|$, a multiple of a norm, or a declared high-vorticity scale. If $\kappa(t)$ changes in time, then the high-vorticity region

$$\Omega_\kappa(t)$$

moves even if the underlying vorticity field changes smoothly.

This can create artificial threshold-crossing events. A complement estimate must distinguish real movement of stretching-active support from motion caused by the threshold itself.

A robust formulation should either:

(i) use fixed thresholds where appropriate;

(ii) control the time variation of $\kappa(t)$;

(iii) use smooth weights with controlled time dependence;

(iv) or include threshold-motion terms in the complement budget.

If threshold motion creates nonsummable flicker constants, the route is not controlled by complement accounting.

7.15 Complement contribution

The low-vorticity complement contribution is the part of the channel remainder assigned to positive stretching outside the selected high-vorticity region or inside the complement of a smooth weight:

$$R_{\text{low}}(t).$$

In sharp-threshold form, one may write schematically

$$R_{\text{low}}(t) = \int_{\Omega\setminus\Omega_\kappa(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V.$$

In smooth-weight form,

$$R_{\text{low}}(t) = \int_{\Omega} (1 - W_\kappa(|\omega|))|\omega|^2 a^+(x,t)\ \mathrm{d}V.$$

If a transition band is treated separately, the contribution may be decomposed as

$$R_{\text{low}}(t) = R_{\text{below}}(t) + R_{\text{band}}(t),$$

where R_{below} lies clearly below the high-vorticity region and R_{band} lies near the threshold boundary.

The exact decomposition is less important than the accounting principle: no positive-stretching contribution may disappear because of a threshold choice.

7.16 Definition of low-vorticity complement channel

We now give the working definition.

Definition 7.1 (Low-vorticity complement channel) *Let u be a smooth solution of the three-dimensional incompressible Navier–Stokes equations on*

$$I = [t_0, t_1].$$

Let $\kappa(t)$ be a declared high-vorticity threshold, or let $W_\kappa(|\omega|)$ be a declared smooth high-vorticity weight.

The low-vorticity complement channel $R_{\mathrm{low}}(I)$ is active on I if positive stretching is non-negligibly carried by one or more of the following:

(i) the sharp complement

$$\Omega \setminus \Omega_\kappa(t);$$

(ii) the complement of a smooth high-vorticity weight

$$1 - W_\kappa(|\omega|);$$

(iii) a threshold transition band;

(iv) a threshold-flickering support whose cumulative positive stretching is significant;

(v) broad low-intensity support outside the selected high-vorticity region;

(vi) scale-local or fragmented support whose main classification issue is threshold or complement assignment.

The channel is active only for the contribution not already assigned entirely to primary entry, coherent aligned-patch support, transition-layer support, fragmentation, scale-local transfer, or residual pathology. Any overlap with those channels must be handled by a partition, bounded-overlap budget, stopping-time reassignment, smooth threshold split, or equivalent no-double-counting rule.

This definition makes R_{low} the threshold-complement channel. It is not a claim that low-vorticity stretching is harmless. It is the ordinary channel that records stretching missed by high-vorticity masks.

7.17 Complement exits

Once complement behavior is active, several outcomes are possible.

First, the contribution may be directly absorbable:

$$R_{\mathrm{low}}(t) \le \delta_{\mathrm{low}} D(t) + C_{\mathrm{low}} E_\omega(t).$$

Second, it may be controlled in subinterval-stable integrated form:

$$\int_{t_0}^{t} R_{\text{low}}(s)\ \mathrm{d}s \le \delta_{\text{low}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\text{low}} \int_{t_0}^{t} E_{\omega}(s)\ \mathrm{d}s + C_{0,\text{low}}$$

for every $t \in I$.

Third, it may become lower-order through enstrophy control, weak alignment, finite threshold-band lifetime, or smooth-weight averaging.

Fourth, it may exit to scale-local control:

$$R_{\text{low}} \Longrightarrow R_{\text{scale}},$$

if the complement contribution is mainly a filtered-scale effect.

Fifth, it may exit to residual pathology:

$$R_{\text{low}} \Longrightarrow R_{\text{path}},$$

if threshold flicker, broad complement stretching, or transition-band support remains dangerous and non-absorbable.

7.18 What complement geometry does not prove

This section defines complement and threshold-flicker geometry. It does not prove complement control.

In particular, it does not prove that complement stretching is always lower-order. It does not prove that threshold-band stretching is absorbable. It does not prove that smooth weights always remove threshold instability. It does not prove that threshold flicker has summable constants. It does not prove that finite-budget flicker control is automatic. It does not prove that broad low-intensity complement support is harmless. It does not prove that every complement route can be controlled without exhausting the dissipation margin.

Those are control questions. They are stated as theorem targets in Section 8.

7.19 Summary

The low-vorticity complement channel R_{low} is activated when positive stretching lies outside the selected high-vorticity region, near a threshold boundary, in the complement of a smooth high-vorticity weight, or in threshold-flickering support. The channel prevents cutoff choices from hiding dangerous stretching.

Complement geometry includes sharp masks, smooth weights, threshold bands, broad low-intensity support, threshold flicker, finite-crossing or summable-flicker requirements, scale-local complement effects, fragmented complement support, and threshold-motion issues. The next section formulates the control targets for R_{low}.

8 Complement Control Targets

The previous section defined low-vorticity complement and threshold-flicker geometry. This section formulates the control targets for the complement channel

$$R_{\text{low}}.$$

The goal is not to prove unconditional complement control in this paper. The goal is to state what must be shown for positive stretching outside a selected high-vorticity mask, near a threshold boundary, or inside a smooth-weight complement to become absorbable, lower-order, scale-local, or residual pathological.

The central question is:

> If positive stretching slips through a high-vorticity threshold, can it still be controlled?

The favorable answer is that complement stretching is lower-order, absorbed by dissipation and enstrophy, controlled by smooth threshold splitting, confined to a finite threshold band, controlled by finite-budget flicker accounting, or reassigned to scale-local control. The dangerous answer is that complement stretching remains significant while avoiding high-vorticity control, complement estimates, threshold-flicker summability, finite-budget threshold accounting, and residual pathological assignment.

8.1 The complement obstruction

The complement channel is obstructive when

$$R_{\text{low}}(t)$$

is non-negligible relative to the channel budget. In sharp-threshold form, this means

$$R_{\text{low}}(t) \sim \int_{\Omega \setminus \Omega_\kappa(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V$$

remains significant. In smooth-weight form, it means

$$R_{\text{low}}(t) \sim \int_{\Omega} (1 - W_\kappa(|\omega|))|\omega|^2 a^+(x,t)\ \mathrm{d}V$$

remains significant.

The strongest complement obstruction has the following features:

(i) the contribution lies outside the selected high-vorticity region or inside a threshold transition band;

(ii) positive strain alignment remains significant;

(iii) the contribution is not lower-order in enstrophy;

(iv) threshold flicker creates nonsummable cumulative stretching;

(v) smooth threshold weights do not stabilize the classification;

(vi) finite-budget flicker control fails;

(vii) the route does not become scale-local or residual pathology in a controlled way.

Such a route would mean that high-vorticity control misses an ordinary positive-stretching contribution that still matters for enstrophy growth.

8.2 Pointwise complement target

The strongest useful target is a pointwise estimate:

$$R_{\mathrm{low}}(t) \le \delta_{\mathrm{low}} D(t) + C_{\mathrm{low}} E_\omega(t).$$

Here $\delta_{\mathrm{low}} \ge 0$ is the dissipation fraction consumed by the complement channel, and $C_{\mathrm{low}} \ge 0$ is a lower-order enstrophy coefficient.

This estimate is useful for closure only if the total margin remains positive:

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} < 1.$$

Equivalently,

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1.$$

Thus complement control is useful only if δ_{low} is small enough to leave reserve for residual pathology and final coefficient recovery.

8.3 Integrated complement target

Complement stretching may be threshold-flickering, moving, broad, or intermittent. A pointwise estimate may be too strong in such cases. The appropriate target is then a subinterval-stable integrated estimate:

$$\int_{t_0}^{t} R_{\mathrm{low}}(s)\ \mathrm{d}s \le \delta_{\mathrm{low}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\mathrm{low}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\mathrm{low}}$$

for every

$$t \in I = [t_0, t_1].$$

If complement control is proved on threshold-flicker intervals

$$I_m = [\tau_m, \tau_{m+1}],$$

then the interval constants must be summable:

$$\sum_m C_{0,m}^{\mathrm{low}} < \infty.$$

Otherwise, infinitely many individually controlled threshold crossings may accumulate into an uncontrolled contribution.

This is the complement-channel version of the burst-summability requirement in Paper 150K [27].

8.4 Lower-order enstrophy route

The complement region has a useful sharp-threshold property:

$$|\omega(x,t)| \le \kappa(t) \qquad \text{on } \Omega \setminus \Omega_\kappa(t).$$

Therefore, complement stretching may be lower-order if the positive alignment factor a^+ can be controlled.

A schematic lower-order route is

$$R_{\mathrm{low}}(t) \le C_{\mathrm{low}} E_\omega(t).$$

More generally, one may seek

$$R_{\mathrm{low}}(t) \le \delta_{\mathrm{low}} D(t) + C_{\mathrm{low}} E_\omega(t)$$

with small δ_{low}.

The main analytic burden is that

$$a^+(x,t) = \max\{n_i S_{ij} n_j, 0\}$$

depends on the strain field, which is nonlocal through the velocity and incompressibility constraint. Thus $|\omega| \le \kappa$ alone does not control R_{low}. Complement control requires an estimate on strain alignment, lower-order strain norms, or the time-integrated contribution.

8.5 Smooth-threshold route

Smooth threshold weights prevent cutoff loss. Let

$$W_\kappa(|\omega|)$$

be a smooth high-vorticity weight. Then

$$W_\kappa(|\omega|) + (1 - W_\kappa(|\omega|)) = 1.$$

Applied to positive stretching,

$$\int_\Omega |\omega|^2 a^+ \, \mathrm{d}V = \int_\Omega W_\kappa(|\omega|)|\omega|^2 a^+ \, \mathrm{d}V + \int_\Omega (1 - W_\kappa(|\omega|))|\omega|^2 a^+ \, \mathrm{d}V.$$

The smooth-threshold control target is to show that the complement-weighted term satisfies

$$\int_\Omega (1 - W_\kappa(|\omega|))|\omega|^2 a^+ \, \mathrm{d}V \le \delta_{\mathrm{low}} D(t) + C_{\mathrm{low}} E_\omega(t),$$

or a subinterval-stable integrated analogue.

This route is especially useful when a sharp threshold creates artificial flicker or unstable classification. Smooth weights spread the transition over a band and make the complement assignment more robust.

8.6 Threshold-band control route

A dangerous contribution may concentrate near the threshold boundary. Let

$$B_\kappa(t) = \{x \in \Omega : \kappa_-(t) \le |\omega(x,t)| \le \kappa_+(t)\}$$

be a threshold band. The positive stretching in the band is

$$P^+_{B_\kappa}(t) = \int_{B_\kappa(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V.$$

A threshold-band control target is

$$P^+_{B_\kappa}(t) \le \delta_{\mathrm{low}} D(t) + C_{\mathrm{low}} E_\omega(t),$$

or

$$\int_{t_0}^t P^+_{B_\kappa}(s)\ \mathrm{d}s \le \delta_{\mathrm{low}} \int_{t_0}^t D(s)\ \mathrm{d}s + C_{\mathrm{low}} \int_{t_0}^t E_\omega(s)\ \mathrm{d}s + C_{0,B}$$

for every $t \in I$.

The difficult case is threshold-band stretching that remains significant while the band is thin, flickering, or moving. Such a route must either be controlled by smooth weights, assigned to scale-local behavior, or moved to residual pathology.

8.7 Threshold-flicker summability route

Threshold flicker occurs when a stretching-active support repeatedly crosses the high-vorticity boundary or smooth transition band. Let

$$I_m = [\tau_m, \tau_{m+1}]$$

be threshold-flicker intervals. The cumulative complement contribution is

$$\sum_m \int_{I_m} R_{\mathrm{low}}(t)\ \mathrm{d}t.$$

A threshold-flicker control target is

$$\sum_m \int_{I_m} R_{\mathrm{low}}(t)\ \mathrm{d}t \le \delta_{\mathrm{low}} \sum_m \int_{I_m} D(t)\ \mathrm{d}t + C_{\mathrm{low}} \sum_m \int_{I_m} E_\omega(t)\ \mathrm{d}t + \sum_m C^{\mathrm{low}}_{0,m}.$$

This estimate is useful only if

$$\sum_m C^{\mathrm{low}}_{0,m} < \infty.$$

If infinitely many threshold crossings occur in finite time and the constants are nonsummable, then complement control fails. The route becomes a residual threshold pathology:

$$R_{\mathrm{low}} \Longrightarrow R_{\mathrm{path}}.$$

8.8 Finite-budget flicker control

A useful way to control threshold flicker is to tie each significant crossing to a finite budget. Let

$$B_{\text{flicker}}(t) \geq 0$$

be a budget density measuring threshold-band cost, alignment-loss cost, threshold-motion cost, or dissipation associated with flicker. If

$$\int_I B_{\text{flicker}}(t)\ \mathrm{d}t < \infty$$

and each significant flicker event consumes at least $b_* > 0$, then only finitely many significant flicker events can occur:

$$N_{\text{flicker}} \leq \frac{1}{b_*} \int_I B_{\text{flicker}}(t)\ \mathrm{d}t.$$

If there are infinitely many flicker events, then either their individual costs must tend to zero or the total budget diverges. A theorem-level complement estimate must therefore prove finite crossing, summable crossing cost, or direct integrated absorbability.

This prevents threshold flicker from becoming an unbounded temporal escape route.

8.9 Broad complement support route

Complement stretching may be broad and weak rather than localized. A large region outside the high-vorticity mask may carry weak but positive alignment. The contribution

$$\int_{\Omega \setminus \Omega_\kappa(t)} |\omega|^2 a^+\ \mathrm{d}V$$

may be significant because of large measure rather than high local intensity.

A broad complement support estimate may use the fact that $|\omega| \leq \kappa(t)$ on the complement, together with bounds on the positive alignment factor. A schematic target is

$$\int_{\Omega \setminus \Omega_\kappa(t)} |\omega|^2 a^+\ \mathrm{d}V \leq C_{\text{broad}} E_\omega(t),$$

or, if strain alignment requires dissipation input,

$$\int_{\Omega \setminus \Omega_\kappa(t)} |\omega|^2 a^+\ \mathrm{d}V \leq \delta_{\text{low}} D(t) + C_{\text{broad}} E_\omega(t).$$

The dangerous case is broad complement support with persistent positive strain alignment that is not lower-order and not absorbable.

8.10 Threshold-motion control route

If the threshold $\kappa(t)$ moves in time, then the high-vorticity region

$$\Omega_\kappa(t) = \{x : |\omega(x,t)| > \kappa(t)\}$$

may change even when the underlying support changes smoothly. This can create artificial threshold crossings.

A threshold-motion control route requires one of the following:

(i) $\kappa(t)$ is fixed on the relevant interval;

(ii) $\kappa(t)$ varies with controlled regularity;

(iii) smooth weights absorb the threshold-motion error;

(iv) threshold-motion terms are included in the complement budget;

(v) or the motion-induced flicker has summable constants.

A schematic threshold-motion estimate is

$$R_{\mathrm{motion}}(t) \leq \delta_{\mathrm{low}} D(t) + C_{\mathrm{motion}} E_{\omega}(t),$$

or an integrated estimate with controlled constants.

If threshold motion creates nonsummable flicker, the route is not complement-controlled.

8.11 Complement-to-scale route

A complement route may become scale-local. For example, positive stretching may lie outside the full-field high-vorticity mask but become visible at a filtered scale:

$$|\omega(x,t)| \leq \kappa(t), \qquad |\omega_{\ell}(x,t)| > \kappa_{\ell}(t).$$

In this case, the route exits to

$$R_{\mathrm{scale}}.$$

The channel transition is

$$R_{\mathrm{low}} \Longrightarrow R_{\mathrm{scale}}.$$

This is useful when the complement contribution is not inherently low-order, but its structure becomes visible and potentially controllable after filtering.

The contribution must be reassigned without double-counting. It should not remain fully charged to R_{low} and R_{scale} unless an overlap coefficient is included in the margin.

8.12 Complement-to-pathology route

If complement stretching remains significant while avoiding pointwise absorbability, integrated absorbability, lower-order control, smooth-threshold stabilization, threshold-band control, flicker summability, finite-budget flicker control, broad-support control, and scale-local classification, then it exits to residual pathology:

$$R_{\mathrm{low}} \Longrightarrow R_{\mathrm{path}}.$$

This is the named threshold-complement failure. It may involve:

(i) nonsummable threshold flicker;

(ii) positive stretching concentrated in a moving threshold band;

(iii) broad complement support with persistent positive strain alignment;

(iv) threshold-sensitive classification not stabilized by smooth weights;

(v) complement support that is neither lower-order nor scale-local;

(vi) threshold-motion errors that accumulate without control.

Such a route remains an obstruction until the pathological-channel layer reduces or absorbs it.

8.13 Conditional complement control alternative

We now state the bridge target for complement control.

Hypothesis 8.1 (Complement control alternative) *Let $R_{\mathrm{low}}(I)$ be active on a smooth interval*

$$I = [t_0, t_1].$$

Then at least one of the following alternatives holds:

(i) ***Pointwise absorbability:***

$$R_{\mathrm{low}}(t) \leq \delta_{\mathrm{low}} D(t) + C_{\mathrm{low}} E_\omega(t)$$

on I.

(ii) ***Integrated absorbability:***

$$\int_{t_0}^{t} R_{\mathrm{low}}(s)\ \mathrm{d}s \leq \delta_{\mathrm{low}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\mathrm{low}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\mathrm{low}}$$

for every $t \in I$.

(iii) ***Lower-order enstrophy control:*** *complement stretching is bounded by lower-order enstrophy terms.*

(iv) ***Smooth-threshold control:*** *a smooth high-vorticity weight stabilizes the split and controls the complement-weighted contribution.*

(v) ***Threshold-band control:*** *positive stretching in the threshold band is absorbable or lower-order.*

(vi) ***Threshold-flicker summability:*** *threshold-crossing intervals have summable residual constants and controlled cumulative contribution.*

(vii) ***Finite-budget flicker control:*** *repeated threshold crossings consume a finite budget, have summable residual constants, or produce directly absorbable cumulative stretching.*

(viii) ***Broad-support control:*** *broad low-intensity complement stretching is lower-order or absorbable.*

(ix) ***Threshold-motion control:*** *time variation of the threshold contributes only controlled terms.*

(x) ***Scale-local exit:***

$$R_{\mathrm{scale}}(I)$$

activates.

(xi) ***Residual pathological exit:***

$$R_{\mathrm{path}}(I)$$

activates.

This hypothesis is not an unconditional theorem. It is the bridge alternative needed for complement stretching to stop being an independent ordinary-channel obstruction.

8.14 Margin condition for complement control

If complement control consumes coefficient δ_{low}, then the final Paper 150J [26] assembly must satisfy

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} < 1.$$

Equivalently,

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1.$$

A useful complement-control estimate should leave reserve for residual pathology and final coefficient recovery:

$$\delta_{\mathrm{low}} \le (1 - \theta - \delta_{\mathrm{coh}} - \delta_{\mathrm{frag}} - \delta_{\mathrm{scale}}) - \delta_{\mathrm{reserve,low}},$$

where

$$\delta_{\mathrm{reserve,low}} > 0.$$

If the best complement estimate satisfies

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} \ge 1,$$

then complement stretching may be visible and estimated, but it is too expensive to close the final margin.

8.15 No double-counting in complement exits

Complement control overlaps naturally with fragmentation and scale-locality. A fragmented support may lie in the complement. A filtered support may move across a threshold. A threshold band may contain many components.

Therefore, a complement-channel assignment must use an explicit accounting rule:

(i) sharp high-vorticity and complement splitting;

(ii) smooth high-vorticity weights;

(iii) transition-band weights;

(iv) stopping-time threshold-crossing intervals;

(v) bounded-overlap channel weights;

(vi) finite-budget flicker accounting;

(vii) or another no-double-counting device.

If a complement route exits to R_{scale}, the contribution should not remain fully counted in R_{low}. If a fragmented complement route is assigned partly to R_{frag}, the overlap must be charged. If a route exits to R_{path}, the independent complement contribution should be removed or overlap-charged.

This bookkeeping is necessary for the final margin condition to have mathematical meaning.

8.16 What complement control would prove

If the complement control alternative holds and the coefficient δ_{low} fits inside the available margin, then complement stretching does not remain an independent ordinary-channel obstruction. Every active complement route is either:

(i) absorbed pointwise;

(ii) absorbed in subinterval-stable integrated form;

(iii) controlled as lower-order enstrophy;

(iv) controlled by smooth threshold splitting;

(v) confined to an absorbable threshold band;

(vi) controlled through threshold-flicker summability;

(vii) controlled by finite-budget flicker accounting;

(viii) controlled as broad lower-order support;

(ix) controlled under threshold motion;

(x) reassigned to scale-local control;

(xi) or assigned to residual pathology.

This is progress even when complement control exits to another channel. It means threshold leakage has been named and placed into the dependency map rather than hidden by a cutoff.

8.17 What this section does not prove

This section does not prove the complement control alternative unconditionally. It does not prove that all complement stretching is lower-order. It does not prove that smooth thresholds always stabilize classification. It does not prove that threshold-band stretching is always absorbable. It does not prove that threshold flicker always has summable constants. It does not prove that finite-budget flicker control is automatic. It does not prove that broad low-intensity complement support is harmless. It does not prove that threshold motion is always controlled.

The section states the theorem target. Later analytic work must prove the required estimates with constants sharp enough to preserve the final margin.

8.18 Summary

The complement channel is controlled if positive stretching outside the high-vorticity mask, in a smooth-weight complement, or inside a threshold band is absorbed by dissipation and lower-order enstrophy, controlled in subinterval-stable integrated form, made lower-order, stabilized by smooth thresholds, controlled through threshold-flicker summability, controlled by finite-budget flicker accounting, or reassigned to R_{scale} or R_{path}.

The central estimate is

$$R_{\mathrm{low}}(t) \leq \delta_{\mathrm{low}} D(t) + C_{\mathrm{low}} E_\omega(t),$$

or its subinterval-stable integrated analogue. The estimate is useful only if

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} < 1$$

remains positive.

The next section assembles the fragmentation, scale-local, and complement alternatives into the conditional remaining-ordinary-channel control theorem.

9 Conditional Remaining-Ordinary-Channel Control Theorem

The previous sections defined and formulated control targets for the three remaining ordinary channels:

$$R_{\mathrm{frag}}, \qquad R_{\mathrm{scale}}, \qquad R_{\mathrm{low}}.$$

This section assembles those alternatives into a conditional remaining-ordinary-channel control theorem.

The theorem is conditional. It does not prove unconditional Navier–Stokes regularity. It states that if fragmentation, scale-local visibility, and low-vorticity complement stretching satisfy the control alternatives stated above, and if the resulting coefficients preserve the Paper 150J [26] margin, then these three ordinary channels do not remain independent obstructions to the final assembly.

9.1 Remaining ordinary-channel contribution

Define the remaining ordinary-channel contribution by

$$R_{\text{rem,ord}}(t) = R_{\text{frag}}(t) + R_{\text{scale}}(t) + R_{\text{low}}(t).$$

This is the ordinary-channel contribution left after coherent aligned-patch and transition-layer control from Paper 150M [29].

The desired pointwise estimate is

$$R_{\text{rem,ord}}(t) \leq \delta_{\text{rem,ord}} D(t) + C_{\text{rem,ord}} E_\omega(t),$$

where

$$\delta_{\text{rem,ord}} = \delta_{\text{frag}} + \delta_{\text{scale}} + \delta_{\text{low}}$$

and

$$C_{\text{rem,ord}} = C_{\text{frag}} + C_{\text{scale}} + C_{\text{low}}.$$

The corresponding integrated estimate is

$$\int_{t_0}^{t} R_{\text{rem,ord}}(s)\ \mathrm{d}s \leq \delta_{\text{rem,ord}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\text{rem,ord}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\text{rem,ord}}$$

for every

$$t \in I = [t_0, t_1].$$

The subinterval condition is essential. It ensures that integrated ordinary-channel control does not hide an intermediate enstrophy spike.

9.2 The ordinary-channel alternatives

Paper 150N uses three channel alternatives.

First, the fragmentation control alternative states that if $R_{\text{frag}}(I)$ is active, then the fragmented route is pointwise absorbable, integrated absorbable, paid for by interface, reconnection, or separation cost, made lower-order by alignment loss or finite lifetime, or exits to R_{scale}, R_{low}, or R_{path}.

Second, the scale-local control alternative states that if $R_{\text{scale}}(I)$ is active, then the scale-local route is pointwise absorbable, integrated absorbable, paid for by high-frequency, transfer, scale-boundary, resolved-scale, or finite-scale-budget cost, made lower-order by alignment loss or finite lifetime, or exits to R_{low} or R_{path}.

Third, the complement control alternative states that if $R_{\text{low}}(I)$ is active, then the complement route is pointwise absorbable, integrated absorbable, lower-order, stabilized by smooth threshold splitting, controlled in a threshold band, controlled by threshold-flicker summability, controlled by finite-budget flicker accounting, controlled by broad-support estimates or threshold-motion estimates, or exits to R_{scale} or R_{path}.

These alternatives can be summarized schematically as

$$R_{\text{frag}} \Longrightarrow \text{cost, lower-order behavior, } R_{\text{scale}},\ R_{\text{low}},\ \text{or } R_{\text{path}},$$

$$R_{\mathrm{scale}} \Longrightarrow \text{cost, lower-order behavior, } R_{\mathrm{low}}, \text{ or } R_{\mathrm{path}},$$

and

$$R_{\mathrm{low}} \Longrightarrow \text{cost, lower-order behavior, } R_{\mathrm{scale}}, \text{ or } R_{\mathrm{path}}.$$

Thus the remaining ordinary channels either become absorbable, become lower-order, or pass the obstruction to a named downstream channel.

9.3 Hypotheses

The theorem uses the following hypotheses.

Hypothesis 9.1 (Fragmentation alternative) *If $R_{\mathrm{frag}}(I)$ is active on a smooth interval*

$$I = [t_0, t_1],$$

then one of the alternatives in Theorem 4.1 holds: pointwise absorbability, integrated absorbability, interface-cost control, reconnection-cost control, alignment loss, finite component lifetime, separation weakening, scale-local exit, threshold/complement exit, or residual pathological exit.

Hypothesis 9.2 (Scale-local alternative) *If $R_{\mathrm{scale}}(I)$ is active on I, then one of the alternatives in Theorem 6.1 holds: pointwise absorbability, integrated absorbability, high-frequency cost, scale-transfer cost, scale-boundary cost, alignment loss, finite scale-local lifetime, resolved-scale accounting, finite scale-budget control, complement exit, or residual pathological exit.*

Hypothesis 9.3 (Complement alternative) *If $R_{\mathrm{low}}(I)$ is active on I, then one of the alternatives in Theorem 8.1 holds: pointwise absorbability, integrated absorbability, lower-order enstrophy control, smooth-threshold control, threshold-band control, threshold-flicker summability, finite-budget flicker control, broad-support control, threshold-motion control, scale-local exit, or residual pathological exit.*

Hypothesis 9.4 (No double-counting among remaining ordinary channels) *All channel transitions and overlaps among*

$$R_{\mathrm{frag}}, \qquad R_{\mathrm{scale}}, \qquad R_{\mathrm{low}}, \qquad R_{\mathrm{path}}$$

are handled by a measurable partition, partition of unity, bounded-overlap budget, stopping-time reassignment, scale decomposition, smooth threshold split, finite-budget event accounting, or equivalent accounting device. No positive-stretching contribution is counted simultaneously in multiple channels unless the overlap is explicitly charged to the dissipation margin.

Hypothesis 9.5 (Subinterval-stable integrated estimates) *Every integrated estimate used for R_{frag}, R_{scale}, or R_{low} is stable on subintervals or on stopping-time partial sums that control every intermediate time. In particular, any estimate of the form*

$$\int_{t_0}^{t} \mathbb{R}_j(s)\,\mathrm{d}s \le \delta_j \int_{t_0}^{t} D(s)\,\mathrm{d}s + C_j \int_{t_0}^{t} E_\omega(s)\,\mathrm{d}s + C_{0,j}$$

must hold for every $t \in I$, or through an equivalent stopping-time construction with summable residual constants.

Hypothesis 9.6 (Scale-density and flicker-summability discipline) *The declared scale family has controlled overlap and finite scale budget:*

$$\delta_{\text{scale,eff}} < \infty.$$

If a countable or continuum scale family is used, the corresponding coefficient sum or coefficient integral is finite.

In addition, all threshold-flicker, fragmentation-burst, scale-transfer, and reconnection-event decompositions used in the estimates have summable residual constants or are controlled by a finite budget. In particular,

$$\sum_m C_{0,m}^{\text{frag}} < \infty, \qquad \sum_m C_{0,m}^{\text{scale}} < \infty, \qquad \sum_m C_{0,m}^{\text{low}} < \infty,$$

whenever those constants appear in stopping-time estimates.

Hypothesis 9.7 (Remaining ordinary-channel margin compatibility) *The remaining ordinary-channel coefficients satisfy*

$$\delta_{\text{rem,ord}} = \delta_{\text{frag}} + \delta_{\text{scale}} + \delta_{\text{low}}$$

after any overlap, scale-budget, reconnection, and finite-event adjustments, and the full Paper 150J margin remains positive:

$$\theta + \delta_{\text{coh}} + \delta_{\text{rem,ord}} + \delta_{\text{path}} < 1.$$

If an ordinary route exits completely to R_{path}, then its contribution is not also counted independently in R_{frag}, R_{scale}, or R_{low}, except through an explicitly charged overlap rule.

9.4 Main theorem

Theorem 9.8 (Conditional remaining-ordinary-channel control theorem) *Let u be a smooth solution of the three-dimensional incompressible Navier–Stokes equations on*

$$I = [t_0, t_1] \subset [0, T].$$

Assume the fragmentation alternative, the scale-local alternative, the complement alternative, no double-counting among remaining ordinary channels, subinterval-stable integrated estimates, scale-density and flicker-summability discipline, and remaining ordinary-channel margin compatibility.

Then the remaining ordinary channels

$$R_{\text{frag}}, \qquad R_{\text{scale}}, \qquad R_{\text{low}}$$

do not remain independent ordinary-channel obstructions to the Paper 150J assembly. More precisely, every active remaining ordinary-channel contribution is either:

(i) absorbed by dissipation and lower-order enstrophy with coefficient included in $\delta_{\text{rem,ord}}$;

(ii) controlled in subinterval-stable integrated form with controlled residual constants;

(iii) made lower-order by alignment loss, finite lifetime, threshold smoothing, broad-support control, or related lower-order behavior;

(iv) paid for by interface, reconnection, separation, high-frequency, scale-transfer, scale-boundary, finite-scale-budget, threshold-band, threshold-flicker, or threshold-motion cost;

(v) reassigned to another remaining ordinary channel without double-counting;

(vi) or reassigned to residual pathology R_{path} as a named downstream obstruction.

If all active contributions in $R_{\text{frag}} + R_{\text{scale}} + R_{\text{low}}$ are absorbed or made lower-order rather than exiting to R_{path}, then the combined remaining ordinary estimate

$$R_{\text{rem,ord}}(t) \le \delta_{\text{rem,ord}} D(t) + C_{\text{rem,ord}} E_\omega(t)$$

holds in pointwise or subinterval-stable integrated form, with $\delta_{\text{rem,ord}}$ entering the Paper 150J margin.

9.5 Proof

Assume first that $R_{\text{frag}}(I)$ is active. By Theorem 9.1, one of the fragmentation alternatives holds.

If pointwise absorbability holds, then

$$R_{\text{frag}}(t) \le \delta_{\text{frag}} D(t) + C_{\text{frag}} E_\omega(t).$$

If integrated absorbability holds, then

$$\int_{t_0}^{t} R_{\text{frag}}(s)\ \mathrm{d}s \le \delta_{\text{frag}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\text{frag}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\text{frag}}$$

for every $t \in I$, or through an equivalent stopping-time partial-sum estimate. In either case, R_{frag} is absorbable in the sense required by the Paper 150K [27] accounting lemmas.

If interface-cost control, reconnection-cost control, alignment loss, finite component lifetime, or separation weakening holds, then the fragmented contribution is either paid for by dissipation and lower-order enstrophy, has summable event cost, returns to a coherent or scale-local classification, or becomes lower-order. It therefore does not remain an independent fragmentation obstruction.

If the route exits to R_{scale}, R_{low}, or R_{path}, then Theorem 9.4 assigns the contribution to the corresponding downstream channel by a partition, stopping-time reassignment, bounded-overlap rule, or equivalent accounting device. It is not counted independently as R_{frag} unless the overlap is explicitly charged.

Now assume $R_{\text{scale}}(I)$ is active. By Theorem 9.2, one of the scale-local alternatives holds.

If pointwise absorbability holds, then

$$R_{\text{scale}}(t) \le \delta_{\text{scale}} D(t) + C_{\text{scale}} E_\omega(t).$$

If integrated absorbability holds, then

$$\int_{t_0}^{t} R_{\text{scale}}(s)\ \mathrm{d}s \le \delta_{\text{scale}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\text{scale}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\text{scale}}$$

for every $t \in I$, or through an equivalent stopping-time construction. If high-frequency cost, scale-transfer cost, scale-boundary cost, alignment loss, finite lifetime, resolved-scale accounting, or finite scale-budget control holds, then the scale-local contribution is either absorbable or lower-order with finite scale-accounting cost. If the route exits to R_{low} or R_{path}, then the no-double-counting hypothesis reassigns the contribution to the corresponding channel.

Finally assume $R_{\text{low}}(I)$ is active. By Theorem 9.3, one of the complement alternatives holds.

If pointwise absorbability holds, then

$$R_{\text{low}}(t) \le \delta_{\text{low}} D(t) + C_{\text{low}} E_\omega(t).$$

If integrated absorbability holds, then

$$\int_{t_0}^{t} R_{\text{low}}(s)\ \mathrm{d}s \le \delta_{\text{low}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\text{low}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\text{low}}$$

for every $t \in I$, or through an equivalent stopping-time construction. If lower-order enstrophy control, smooth-threshold control, threshold-band control, threshold-flicker summability, finite-budget flicker control, broad-support control, or threshold-motion control holds, then the complement contribution is absorbable or lower-order. If the route exits to R_{scale} or R_{path}, then the no-double-counting hypothesis assigns the contribution to the appropriate downstream channel.

By Theorem 9.5, all integrated estimates used above are compatible with the Paper 150K integrated Gronwall and hidden-spike accounting. By Theorem 9.6, scale overlap, scale coefficient sums, threshold-flicker constants, fragmentation-burst constants, scale-transfer constants, and reconnection-event constants are finite or summable whenever they appear. By Theorem 9.7, the combined ordinary coefficient

$$\delta_{\text{rem,ord}} = \delta_{\text{frag}} + \delta_{\text{scale}} + \delta_{\text{low}}$$

is included in the final margin:

$$\theta + \delta_{\text{coh}} + \delta_{\text{rem,ord}} + \delta_{\text{path}} < 1.$$

Therefore the active remaining ordinary-channel contributions are either absorbed with coefficient included in $\delta_{\text{rem,ord}}$, made lower-order, controlled by finite event or scale budgets, or reassigned to a named downstream channel without uncontrolled double-counting. Hence R_{frag}, R_{scale}, and R_{low} do not remain independent ordinary-channel obstructions to the Paper 150J assembly.

If none of the active remaining ordinary-channel contributions exits to R_{path}, then summing the pointwise or integrated estimates yields

$$R_{\text{rem,ord}}(t) \le \delta_{\text{rem,ord}} D(t) + C_{\text{rem,ord}} E_\omega(t)$$

or the corresponding subinterval-stable integrated estimate. This completes the proof.

9.6 Combined pointwise corollary

Corollary 9.9 (Pointwise remaining ordinary-channel estimate) *Assume that all active contributions in*

$$R_{\text{frag}}, \qquad R_{\text{scale}}, \qquad R_{\text{low}}$$

satisfy pointwise estimates

$$R_{\text{frag}}(t) \le \delta_{\text{frag}} D(t) + C_{\text{frag}} E_\omega(t),$$

$$R_{\mathrm{scale}}(t) \le \delta_{\mathrm{scale}} D(t) + C_{\mathrm{scale}} E_\omega(t),$$

and

$$R_{\mathrm{low}}(t) \le \delta_{\mathrm{low}} D(t) + C_{\mathrm{low}} E_\omega(t),$$

with no uncontrolled double-counting and with finite scale-overlap and event-accounting budgets. Then

$$R_{\mathrm{rem,ord}}(t) \le \delta_{\mathrm{rem,ord}} D(t) + C_{\mathrm{rem,ord}} E_\omega(t),$$

where

$$\delta_{\mathrm{rem,ord}} = \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}}$$

and

$$C_{\mathrm{rem,ord}} = C_{\mathrm{frag}} + C_{\mathrm{scale}} + C_{\mathrm{low}}.$$

If

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1,$$

then the remaining ordinary channels fit inside the Paper 150J dissipation margin.

Sum the three pointwise estimates. The no-double-counting hypothesis ensures that the same positive-stretching contribution is not charged repeatedly without an overlap budget. The scale-density and flicker-summability hypothesis ensures that scale and event budgets remain finite. The stated coefficient is the sum of the three ordinary-channel coefficients after any overlap, scale-budget, and event-accounting adjustment.

9.7 Combined integrated corollary

Corollary 9.10 (Subinterval-stable integrated remaining ordinary-channel estimate) *Assume that all active contributions in*

$$R_{\mathrm{frag}}, \qquad R_{\mathrm{scale}}, \qquad R_{\mathrm{low}}$$

satisfy integrated estimates on every subinterval $[t_0, t] \subset I$, or on stopping-time partial sums controlling every intermediate time:

$$\int_{t_0}^{t} R_{\mathrm{frag}}(s)\ \mathrm{d}s \le \delta_{\mathrm{frag}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\mathrm{frag}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\mathrm{frag}},$$

$$\int_{t_0}^{t} R_{\mathrm{scale}}(s)\ \mathrm{d}s \le \delta_{\mathrm{scale}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\mathrm{scale}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\mathrm{scale}},$$

and

$$\int_{t_0}^{t} R_{\mathrm{low}}(s)\ \mathrm{d}s \le \delta_{\mathrm{low}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\mathrm{low}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\mathrm{low}}.$$

Then

$$\int_{t_0}^{t} R_{\mathrm{rem,ord}}(s)\ \mathrm{d}s \le \delta_{\mathrm{rem,ord}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\mathrm{rem,ord}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\mathrm{rem,ord}},$$

where

$$C_{0,\mathrm{rem,ord}} = C_{0,\mathrm{frag}} + C_{0,\mathrm{scale}} + C_{0,\mathrm{low}}$$

after any explicitly charged finite event-budget constants are included. If the constants arise from burst, reconnection, scale-transfer, threshold-flicker, or stopping-time decompositions, the corresponding partial sums must be summable and controlled uniformly over subintervals.

Sum the three integrated estimates over the common interval $[t_0, t]$, or over the stopping-time partial sums up to t. The subinterval-stability hypothesis ensures that the estimate controls every intermediate time. The scale-density and flicker-summability hypothesis ensures that the relevant event constants are finite or summable. The residual constants add linearly after any explicitly charged finite-budget terms are included.

9.8 Bridge status of the theorem

Theorem 9.8 is a dependency theorem. It does not prove the fragmentation, scale-local, or complement estimates from the Navier–Stokes dynamics. It proves that, if those estimates or alternatives are supplied with no double-counting, subinterval stability, finite scale and event budgets, and margin compatibility, then the remaining ordinary channels do not remain independent obstructions.

This distinction is important. The theorem should not be read as saying that fragmentation, scale-locality, or complement behavior is automatically harmless. It says that these channels are harmless only after they are absorbed, made lower-order, or assigned to a downstream channel with the accounting and margin preserved.

9.9 Role of pathological exits

The theorem allows routes to exit to

$$R_{\text{path}}.$$

This does not close the final regularity program. It identifies a downstream dependency.

If a fragmented, scale-local, or complement route exits to R_{path}, then the ordinary-channel layer has succeeded only in naming the residual obstruction. The remaining task is pathological reduction or absorption:

$$R_{\text{path}} \Longrightarrow \text{ordinary-channel reduction or absorbable residual cost.}$$

That task belongs to the pathological-channel layer and any later refinement of it.

Thus Paper 150N should be read as:

$$\text{remaining ordinary channels} \Longrightarrow \text{absorption, lower-order behavior, or named pathology.}$$

It is not a proof that all residual pathology is controlled.

9.10 What the theorem proves

The theorem proves a conditional dependency result. Under the stated alternatives, the remaining ordinary channels

$$R_{\text{frag}}, \qquad R_{\text{scale}}, \qquad R_{\text{low}}$$

cannot remain undefined ordinary-channel obstructions. They are either controlled, reduced to lower-order behavior, controlled by finite scale or event budgets, or reassigned to named downstream channels.

In the favorable case where no active contribution exits to R_{path}, the theorem gives the combined remaining ordinary estimate:

$$R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}} \le \delta_{\mathrm{rem,ord}} D + C_{\mathrm{rem,ord}} E_{\omega}$$

or its subinterval-stable integrated analogue.

This is exactly the estimate needed to move from coherent-channel control in Paper 150M toward pathological refinement and coefficient recovery in later papers.

9.11 What the theorem does not prove

The theorem does not prove the fragmentation, scale-local, or complement alternatives unconditionally. It does not prove that all fragmented supports pay interface cost. It does not prove that all reconnection events pay reconnection cost. It does not prove that every scale-local route pays high-frequency cost. It does not prove that every scale family is complete or has finite coefficient budget. It does not prove that every complement contribution is lower-order. It does not prove that all burst constants are summable. It does not prove that all threshold flicker is controlled. It does not prove that residual pathology is absorbable.

It also does not prove the full Navier–Stokes regularity theorem. Even if the remaining ordinary channels are controlled, the Paper 150J assembly still requires:

(i) visibility or universal entry;

(ii) primary depletion with coefficient $\theta < 1$;

(iii) coherent-channel control from Paper 150M;

(iv) remaining ordinary-channel control from the present paper;

(v) pathological reduction or absorption;

(vi) no double-counting;

(vii) integrated-to-uniform continuation;

(viii) and final margin preservation.

Thus Paper 150N is a bridge-target paper, not a final proof.

9.12 Summary

This section assembled the fragmentation, scale-local, and complement alternatives into the conditional remaining-ordinary-channel control theorem. Under the stated hypotheses,

$$R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}}$$

is either absorbable, controlled in subinterval-stable integrated form, made lower-order, paid for by ordinary costs, controlled by finite scale or event budgets, reassigned among remaining ordinary channels, or moved into residual pathology.

The favorable estimate is

$$R_{\mathrm{rem,ord}}(t) \leq \delta_{\mathrm{rem,ord}} D(t) + C_{\mathrm{rem,ord}} E_\omega(t),$$

or its integrated analogue, with

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1.$$

The next section states falsifiers and failure modes for these remaining ordinary-channel control targets.

10 Falsifiability and Failure Modes

The previous section stated the conditional remaining-ordinary-channel control theorem. This section identifies how the theorem can fail. The purpose is to keep Paper 150N falsifiable. A bridge paper is useful only if its failure modes are explicit and assigned to named mathematical obstructions.

The central target of Paper 150N is

$$R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}} \quad \Longrightarrow \quad \text{absorbable cost, lower-order behavior, or named residual pathology.}$$

A falsifier is therefore a smooth solution, a controlled limiting sequence of smooth solutions, or a resolution-stable numerical candidate in which fragmentation, scale-local visibility, or complement stretching remains significant while avoiding absorbability, lower-order control, channel reassignment, and residual pathological classification.

Numerical evidence alone does not prove or disprove Navier–Stokes regularity. Its role is diagnostic: it may reveal which channel fails, which coefficient exhausts the margin, or which support geometry remains unresolved. A theorem-level failure must ultimately be stated in the classical Navier–Stokes quantities used throughout this paper.

10.1 Failure by persistent fragmentation without interface cost

The most direct fragmentation failure is a fragmented support that preserves positive stretching while paying too little interface or boundary cost.

Let

$$A(t) = \bigcup_{j \in J(t)} A_j(t)$$

be a fragmented stretching-active support. A schematic failure has

$$\sum_j \int_{A_j(t)} |\omega|^2 a^+ \, \mathrm{d}V$$

significant, while

$$\nu \sum_j \int_{N_r(\partial A_j(t))} |\nabla \omega|^2 \, \mathrm{d}V$$

is too small to absorb the contribution.

This would mean that breaking coherent support into many pieces does not force enough magnitude-gradient, directional-gradient, boundary, or interface cost. If the route also avoids alignment loss, finite component lifetime, scale-local classification, complement assignment, and residual pathology, then R_{frag} remains an independent obstruction.

10.2 Failure by reconnection without gradient cost

A sharper fragmentation failure occurs when fragmented components reconnect while preserving positive stretching and paying too little gradient cost. Reconnection can merge, braid, or exchange support among components. If the event preserves alignment and positive stretching while avoiding both interface and reconnection-neighborhood cost, then fragmentation control has not succeeded.

Let $\mathcal{R}_{\mathrm{rec}}(t)$ be an active reconnection region. A schematic reconnection failure has

$$R_{\mathrm{rec}}(t) \quad \text{significant},$$

while

$$C_{\mathrm{rec}}(t) = \nu \int_{N_r(\mathcal{R}_{\mathrm{rec}}(t))} |\nabla\omega|^2 \, \mathrm{d}V$$

is too small to absorb it.

This failure is important because N_{eff}^{+} alone may not detect a reconnection event that temporarily preserves positive stretching while changing topology. A theorem-level fragmentation estimate must therefore show that reconnection either pays cost, returns to a coherent channel, becomes scale-local, enters the complement, or exits to R_{path}.

10.3 Failure by fragmented support with preserved alignment

Fragmentation is favorable only if it weakens coherent positive strain alignment or creates cost. A failure occurs if many components preserve strong positive alignment.

For each component, recall

$$\Pi_j^{+}(t) = \frac{\int_{A_j(t)} |\omega|^2 a^{+} \, \mathrm{d}V}{\int_{A_j(t)} |\omega|^2 |a| \, \mathrm{d}V + \varepsilon}.$$

A fragmented-alignment falsifier has

$$N_{\mathrm{eff}}^{+}(t) \gg 1, \qquad \Pi_j^{+}(t) \approx 1$$

for many positive-stretching-dominant components, while the total fragmented contribution remains significant and interface cost remains too small.

This would show that many pieces can collectively preserve the same dangerous strain alignment that coherent patches preserved before fragmentation.

10.4 Failure by long-lived fragmented components

A fragmented route may fail by persistence. Components may be individually small but survive long enough to generate significant integrated positive stretching.

Let

$$I_j = [\tau_j, \tau_{j+1}]$$

be a component-tracking interval. A lifetime falsifier has

$$\sum_j \int_{I_j} \int_{A_j(t)} |\omega|^2 a^+ \, \mathrm{d}V \, \mathrm{d}t$$

significant, while the component lifetimes are not short relative to local stretching times and the accumulated costs are not absorbable.

This failure matters because fragmentation may reduce instantaneous coherence but still preserve cumulative amplification over many long-lived pieces.

10.5 Failure by nonsummable fragmented bursts

Fragmentation may appear through many short bursts. Each burst may appear individually controlled, but the burst constants may fail to sum.

Let

$$I_m = [\tau_m, \tau_{m+1}]$$

be fragmentation burst intervals. A nonsummable burst falsifier has estimates of the form

$$\int_{I_m} R_{\mathrm{frag}}(t) \, \mathrm{d}t \le \delta_{\mathrm{frag}} \int_{I_m} D(t) \, \mathrm{d}t + C_{\mathrm{frag}} \int_{I_m} E_\omega(t) \, \mathrm{d}t + C_{0,m}^{\mathrm{frag}},$$

but

$$\sum_m C_{0,m}^{\mathrm{frag}} = \infty.$$

Then the burst-level estimates cannot be assembled into a global subinterval-stable bound.

This is a Zeno-type fragmentation failure: infinitely many apparently controlled events accumulate into an uncontrolled total contribution.

10.6 Failure by untrackable moving or reconnecting fragments

Fragments may move, merge, split, or reconnect. A failure occurs if this motion prevents a stable component assignment while cumulative positive stretching remains significant.

A moving-fragment falsifier has

$$\int_I \sum_j \int_{A_j(t)} |\omega|^2 a^+ \, \mathrm{d}V \, \mathrm{d}t$$

significant, while no material, weighted, graph-based, or stopping-time decomposition supplies controlled partial sums.

This would mean that fragmentation is visible in principle but not measurable or assignable in a way compatible with the accounting layer. The remedy would be a stronger component-tracking lemma, a weighted decomposition, or reassignment to residual pathology.

10.7 Failure by scale-local positive stretching without gradient cost

The most direct scale-local failure is filtered positive stretching without corresponding high-frequency or gradient cost.

For a declared active scale ℓ, a schematic failure has

$$\int_\Omega \chi_\ell |\omega_\ell|^2 a_\ell^+ \, \mathrm{d}V$$

significant, while the associated high-frequency or scale-local dissipation

$$D_{\mathrm{hf}}(t)$$

is too small to absorb it.

Such a route would show that filtering reveals organized positive stretching, but the expected small-scale gradient cost does not appear. If the route also avoids finite lifetime, alignment loss, scale-boundary cost, complement assignment, and residual pathology, then R_{scale} remains an independent obstruction.

10.8 Failure by incomplete scale family

A scale-local conclusion can fail if the declared scale family is incomplete. Suppose the dangerous support lies below the smallest declared scale

$$\ell_{\min} = \min_{\ell \in \mathcal{L}} \ell.$$

If the support is missed only because $\ell_{\min}$ is too coarse, then the result is diagnostic incompleteness, not theorem-level control.

The failure mode is:

$$\text{positive stretching remains significant below the declared scale floor,}$$

while no unresolved-scale cost estimate is proved and no residual pathological assignment is made.

This failure does not refute the channel architecture. It shows that the declared scale family was insufficient for the claim being made.

10.9 Failure by divergent scale budget

A scale-local conclusion can also fail when the declared scale family is too redundant or too dense without controlled coefficient accounting. Even if every individual scale satisfies a local estimate, the total scale budget may diverge.

In a countable scale family, the failure has the schematic form

$$\sum_{\ell \in \mathcal{L}} \delta_\ell = \infty.$$

In a continuum scale family, the corresponding failure is

$$\int_{\mathcal{L}} \delta(\ell) \, \mathrm{d}\mu(\ell) = \infty.$$

Such a result is not a valid scale-local closure. It means that the scale family detects structure but consumes an unbounded amount of dissipation budget. The remedy is controlled overlap, a summable scale decomposition, an orthogonal or almost-orthogonal scale partition, or reassignment of the scale-evading contribution to R_{path}.

10.10 Failure by scale transfer with nonsummable cost

Scale-local activity may move across scales through repeated transfer events. Each transfer may appear small, but the accumulated constants may diverge.

A scale-transfer failure has

$$\sum_m C_{0,m}^{\text{scale}} = \infty$$

for scale-transfer or scale-burst intervals, even though each individual event satisfies a local estimate.

This is the scale-local analogue of Zeno burst failure. It means that scale transfer has not been controlled by a finite dissipation, enstrophy, high-frequency, or transfer budget.

10.11 Failure by scale-sensitive classification

A route may change classification under nearby admissible filters. For example, it may appear fragmented at one scale, coherent at another, and complement-supported at a third.

A scale-sensitive classification failure occurs if the qualitative channel assignment changes under small admissible changes of ℓ, G_ℓ, or the scale weights χ_ℓ, while the positive-stretching contribution remains significant.

This does not automatically mean the route is pathological. It means the scale-local diagnostic is unstable. The remedy is to use a robust scale family, bounded-overlap accounting, or a scale-stable channel assignment.

10.12 Failure by complement stretching that is not lower-order

The most direct complement failure is positive stretching outside the high-vorticity region that is not lower-order.

For a sharp threshold,

$$\Omega_\kappa(t) = \{x : |\omega(x,t)| > \kappa(t)\},$$

a complement falsifier has

$$\int_{\Omega\setminus\Omega_\kappa(t)} |\omega|^2 a^+ \, \mathrm{d}V$$

significant, while no estimate of the form

$$R_{\mathrm{low}}(t) \le \delta_{\mathrm{low}} D(t) + C_{\mathrm{low}} E_\omega(t)$$

holds with margin-compatible δ_{low}.

This would mean that dangerous stretching can remain outside the selected high-vorticity region and still materially affect enstrophy growth.

10.13 Failure by broad low-intensity complement support

Complement stretching may be broad and weak rather than localized. A failure occurs if weak positive alignment over a large complement region accumulates significant stretching:

$$\int_I \int_{\Omega\setminus\Omega_\kappa(t)} |\omega|^2 a^+ \, \mathrm{d}V \, \mathrm{d}t$$

while lower-order enstrophy control fails.

This is the complement analogue of wide-area low-intensity danger. It is important because a proof that only controls localized complement support may miss broad distributed positive alignment.

10.14 Failure by threshold-band concentration

A route may concentrate near the threshold boundary. Let

$$B_\kappa(t) = \{x : \kappa_-(t) \le |\omega(x,t)| \le \kappa_+(t)\}.$$

A threshold-band falsifier has

$$\int_{B_\kappa(t)} |\omega|^2 a^+ \, \mathrm{d}V$$

significant, while the band contribution is not absorbable, not lower-order, and not stabilized by smooth threshold weights.

This would show that the proof can fail not in the high-vorticity region or the complement interior, but in the transition band between them.

10.15 Failure by nonsummable threshold flicker

Threshold flicker occurs when active support repeatedly crosses the high-vorticity boundary. A nonsummable threshold-flicker failure has stopping intervals

$$I_m = [\tau_m, \tau_{m+1}]$$

such that

$$\int_{I_m} R_{\mathrm{low}}(t) \, \mathrm{d}t$$

is individually estimated, but

$$\sum_m C^{\text{low}}_{0,m} = \infty.$$

Then infinitely many threshold crossings can accumulate uncontrolled complement contribution in finite time. This failure requires either a stronger smooth-threshold estimate, a finite-budget flicker lemma, or reassignment to R_{path}.

10.16 Failure by absence of finite-budget flicker control

A threshold-flicker sequence may fail even if individual crossings look small. The key obstruction is the absence of a finite budget that each significant crossing consumes.

Let $B_{\text{flicker}}(t) \geq 0$ be a proposed flicker-control budget. A finite-budget failure occurs if either

$$\int_I B_{\text{flicker}}(t)\ \mathrm{d}t = \infty,$$

or if significant crossings do not consume a controlled amount of this budget. In that case, one cannot conclude that the number of significant crossings is finite or that the accumulated constants are summable.

This failure is distinct from ordinary nonsummability. It says that the mechanism intended to force summability is itself missing or too weak.

10.17 Failure by threshold-motion artifacts

If the threshold $\kappa(t)$ varies in time, then the high-vorticity set

$$\Omega_\kappa(t) = \{x : |\omega(x,t)| > \kappa(t)\}$$

can move even if the flow changes smoothly. A failure occurs if this threshold motion creates artificial complement bursts whose residual constants are not controlled.

This is not necessarily a Navier–Stokes obstruction. It may be a threshold-choice obstruction. The remedy is to use fixed thresholds, smooth weights with controlled time dependence, or explicit threshold-motion correction terms.

10.18 Failure by smooth-threshold instability

Smooth thresholds are meant to stabilize cutoff behavior. They fail if nearby smooth weights give incompatible conclusions.

Let

$$W^{(\rho)}_\kappa(|\omega|)$$

be a family of smooth threshold weights. A smooth-threshold instability occurs if

$$\int_\Omega (1 - W^{(\rho)}_\kappa(|\omega|))|\omega|^2 a^+\ \mathrm{d}V$$

changes qualitatively under small admissible changes of ρ, while the positive-stretching contribution remains significant.

Such a failure means that the complement classification is not robust enough for theorem-level accounting.

10.19 Failure by channel overlap and double-counting

The remaining ordinary channels overlap naturally. A route may be fragmented, scale-local, and threshold-sensitive at the same time. If the same positive-stretching contribution is charged fully to multiple channels, the margin budget may be false.

A double-counting failure occurs if one writes

$$R_{\mathrm{rem,ord}} = R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}}$$

but the three terms contain overlapping copies of the same contribution without an overlap factor or partition rule.

The remedy is an exact partition, partition of unity, bounded-overlap constant, stopping-time reassignment, smooth threshold split, finite-budget event accounting, or another explicit accounting device. If overlap is present, the final margin must use the overlap-adjusted coefficient.

10.20 Failure by margin exhaustion

Even if all remaining ordinary channels are visible and estimated, the final theorem can fail if the estimates consume too much dissipation.

The margin condition is

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} < 1.$$

A margin-exhaustion failure occurs if

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} + \delta_{\mathrm{path}} \geq 1.$$

In that case, the channels may be controlled in a formal sense, but not sharply enough to close the enstrophy inequality.

A weaker but still serious version occurs when

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}} < 1$$

but the remaining reserve is too small to handle R_{path}. This is ordinary-channel budget dominance. It means the ordinary channels succeeded locally while leaving insufficient room for residual pathology.

10.21 Failure by hidden integrated spike

Integrated estimates are useful only if they control every intermediate time. A hidden-spike failure occurs if an estimate holds on the full interval

$$[t_0, t_1]$$

but fails on some subinterval

$$[t_0, t] \subset [t_0, t_1].$$

For example, one might have

$$\int_{t_0}^{t_1} R_{\text{rem,ord}}(s)\ \mathrm{d}s$$

controlled, while $E_\omega(t)$ experiences a large intermediate spike. Such an estimate cannot support the Paper 150J [26] assembly unless a separate continuation bridge rules out the spike.

This is why Paper 150N requires subinterval-stable integrated estimates or stopping-time partial sums controlling every intermediate time.

10.22 Failure by downstream pathological overload

The theorem allows ordinary channels to exit to

$$R_{\text{path}}.$$

This is useful only if the pathological-channel layer can later reduce or absorb the residual contribution.

A downstream pathological overload occurs if many failures of R_{frag}, R_{scale}, and R_{low} all exit into R_{path}, producing a pathological contribution too large to control:

$$R_{\text{path}}(t) \not\lesssim \delta_{\text{path}} D(t) + C_{\text{path}} E_\omega(t)$$

with margin-compatible δ_{path}.

This does not invalidate Paper 150N by itself. It identifies the next obstruction. The final assembly still fails unless residual pathology is controlled.

10.23 Failure by numerical overinterpretation

Numerical simulations can help identify fragmentation, reconnection, scale-local structures, complement leakage, threshold flicker, and possible pathological exits. But numerical evidence does not replace analytic estimates.

A numerical overinterpretation failure occurs if one concludes that R_{frag}, R_{scale}, or R_{low} is controlled merely because a simulation shows favorable behavior in tested regimes.

A credible numerical diagnostic must be:

(i) resolution-aware;

(ii) threshold-stable;

(iii) scale-family explicit;

(iv) checked for no double-counting;

(v) tested for subinterval stability;

(vi) tested for finite scale and event budgets;

(vii) and tied to physical Navier–Stokes quantities such as ω, S, a^+, E_ω, D, component structure, reconnection neighborhoods, filtered support, and threshold weights.

Numerical work can guide bridge hypotheses and reveal falsifiers. It cannot substitute for the bridge estimates.

10.24 What would strengthen Paper 150N

Paper 150N would be strengthened by theorem-level estimates showing:

(i) fragmented support pays interface or boundary cost;

(ii) reconnection events either pay enstrophy-gradient cost, return to a coherent channel, become scale-local, or exit to residual pathology;

(iii) fragmented components lose coherent positive strain alignment;

(iv) component lifetimes or burst constants are summable;

(v) moving and reconnecting fragments admit stable tracking;

(vi) scale-local support pays high-frequency or transfer cost;

(vii) declared scale families are complete for the theorem target;

(viii) the declared scale family is fine enough to detect active scale-local support while keeping the scale-overlap coefficient finite;

(ix) unresolved scales become dissipative or residual pathology;

(x) complement stretching is lower-order or absorbable;

(xi) threshold bands are controlled by smooth weights;

(xii) infinitely many threshold crossings either have summable constants or consume a finite control budget;

(xiii) broad low-intensity complement support is lower-order;

(xiv) the combined coefficient $\delta_{\mathrm{rem,ord}}$ preserves reserve for R_{path}.

Each estimate would reduce the size of the remaining ordinary-channel obstruction and make the final Paper 150J assembly more plausible.

10.25 Summary

Paper 150N can fail in named ways. Fragmentation can fail by preserving positive stretching without interface cost, reconnection cost, alignment loss, finite lifetime, or summable bursts. Scale-local control can fail through filtered positive stretching without gradient cost, incomplete scale families, divergent scale budgets, nonsummable scale transfer, or scale-sensitive classification. Complement control can fail through broad low-intensity stretching, threshold-band concentration, nonsummable threshold flicker, absence of finite-budget flicker control, threshold-motion artifacts, or smooth-threshold instability.

The remaining global failures are double-counting, margin exhaustion, hidden integrated spikes, downstream pathological overload, and numerical overinterpretation.

These failure modes do not make the framework vague. They identify exactly what later bridge work must prove or where the ordinary-channel program can break.

11 Relation to Papers 150J, 150K, 150L, and 150M

The previous section stated falsifiers and failure modes for the remaining ordinary-channel control targets. This section explains how Paper 150N fits into the 150-series architecture.

The role of Paper 150N is specific. It does not introduce a new closure principle. It does not replace the conditional assembly of Paper 150J [26], the accounting lemmas of Paper 150K [27], the visibility bridge of Paper 150L [28], or the coherent-channel control targets of Paper 150M [29]. Instead, it handles the ordinary downstream channels left after coherent aligned patches and transition layers have been addressed:

$$R_{\mathrm{frag}}, \qquad R_{\mathrm{scale}}, \qquad R_{\mathrm{low}}.$$

In the bridge sequence, Paper 150M studied coherent structure. Paper 150N studies what happens when that coherent structure breaks, becomes scale-local, or slips across high-vorticity thresholds.

11.1 Relation to Paper 150J: conditional assembly

Paper 150J [26] assembled the conditional enstrophy-bound theorem for the 150-series. In schematic form, the Paper 150J assembly requires:

$$\text{visibility} + \text{channel control} + \text{pathological closure} + \text{margin preservation} + \text{continuation} \Longrightarrow \sup_{t\in I} E_\omega(t) < \infty.$$

The starting point is the primary depleted stretching estimate

$$P(t) \le \theta D(t) + C E_\omega(t) + R_\kappa(t), \qquad 0 \le \theta < 1,$$

together with the channel decomposition

$$R_\kappa = R^+_{\mathrm{patch}} + R_{\mathrm{trans}} + R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}} + R_{\mathrm{path}}.$$

Paper 150N contributes to the channel-control part of this assembly. Specifically, it formulates control alternatives for

$$R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}}.$$

If these remaining ordinary channels satisfy

$$R_{\mathrm{frag}}(t) + R_{\mathrm{scale}}(t) + R_{\mathrm{low}}(t) \le \delta_{\mathrm{rem,ord}} D(t) + C_{\mathrm{rem,ord}} E_\omega(t),$$

or the corresponding subinterval-stable integrated estimate, then Paper 150N supplies the remaining ordinary-channel estimate needed by the Paper 150J margin.

The relevant margin condition is

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1,$$

where δ_{coh} is the coherent-channel coefficient from Paper 150M [29], $\delta_{\mathrm{rem,ord}}$ is the remaining ordinary-channel coefficient from Paper 150N, and δ_{path} is the residual pathological-channel coefficient.

Thus Paper 150N is not a final assembly theorem. It supplies one piece of the Paper 150J dependency map.

11.2 Relation to Paper 150K: accounting after assignment

Paper 150K [27] proved the unconditional accounting lemmas used once channel assignments and estimates are available. Those lemmas include partition-of-unity splitting, bounded-overlap budget inflation, pointwise Gronwall closure, integrated Gronwall closure, burst summability, superlevel and threshold bookkeeping, and pathological-reduction bookkeeping.

Paper 150N relies on those accounting lemmas in several places.

First, fragmentation requires no-double-counting. A fragmented support may overlap with scale-local or complement diagnostics. If the positive-stretching density is assigned by weights

$$\chi_{\mathrm{frag}}, \qquad \chi_{\mathrm{scale}}, \qquad \chi_{\mathrm{low}},$$

then an exact partition satisfies

$$\chi_{\mathrm{frag}} + \chi_{\mathrm{scale}} + \chi_{\mathrm{low}} = 1$$

on the remaining ordinary-channel support. If the weights overlap, the overlap constant must be charged to the margin.

Second, scale-local filtering can show the same structure at more than one scale. Paper 150K's bounded-overlap accounting is needed whenever

$$\sum_{\ell \in \mathcal{L}} \chi_\ell(x,t) > 1.$$

The resulting effective coefficient must include the scale-overlap cost. Paper 150N further requires that scale-family density not silently inflate the budget: finite scale families need finite coefficient sums, while countable or continuum scale families need summable or integrable scale coefficients.

Third, complement control depends on threshold splitting. The smooth-threshold identity

$$W_\kappa(|\omega|) + \big(1 - W_\kappa(|\omega|)\big) = 1$$

is an accounting fact: positive stretching cannot disappear through a cutoff. Paper 150N uses this to keep high-vorticity and complement contributions assigned without loss.

Fourth, moving fragments, reconnecting fragments, scale-local bursts, scale-transfer events, and threshold-flicker intervals require burst or event summability. If estimates are proved on intervals

$$I_m = [\tau_m, \tau_{m+1}],$$

then their constants must satisfy

$$\sum_m C_{0,m} < \infty,$$

or be controlled by a finite budget. This is the Zeno-exclusion accounting principle from Paper 150K applied to the remaining ordinary-channel layer.

Fifth, any integrated estimate used in Paper 150N must be subinterval-stable. A full-interval estimate is not enough if it can hide an intermediate enstrophy spike. Thus estimates of the form

$$\int_{t_0}^{t} \mathbb{R}_j(s)\ \mathrm{d}s \le \delta_j \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_j \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,j}$$

must hold for every $t \in I$, or through stopping-time partial sums controlling all intermediate times.

Thus Paper 150N supplies channel-control targets, while Paper 150K supplies the accounting rules that make those targets usable.

11.3 Relation to Paper 150L: from visibility to remaining ordinary channels

Paper 150L [28] formulated the refined visibility bridge:

$$D_{\mathrm{amp}}(I) \Longrightarrow E_{\mathrm{entry}}(I) \vee C_{\mathrm{chan}}(I).$$

In expanded form, dangerous amplification must become visible as primary entry or as one of the named channels:

$$E_{\mathrm{entry}}, \qquad R^{+}_{\mathrm{patch}}, \qquad R_{\mathrm{trans}}, \qquad R_{\mathrm{frag}}, \qquad R_{\mathrm{scale}}, \qquad R_{\mathrm{low}}, \qquad R_{\mathrm{path}}.$$

Paper 150L answers the visibility question:

Where is dangerous positive stretching being preserved?

Paper 150N answers a later control question:

If the visible route is fragmented, scale-local, or threshold-complementary, can it be controlled or reclassified?

Thus Paper 150N begins after visibility. It assumes that the remaining ordinary channels may have become active, then asks whether they are absorbable, lower-order, finite-budget controlled, or residual.

The connection can be summarized as

Paper 150L: danger becomes visible,

$$\text{Paper 150N: remaining ordinary visible routes are controlled or passed downstream.}$$

This separation is essential. Visibility is not control. A fragmented route may be visible and still uncontrolled. A scale-local route may be visible and still too expensive. A complement route may be visible and still fail the margin. Paper 150N studies these control problems without replacing the visibility bridge.

11.4 Relation to Paper 150M: downstream exits from coherent channels

Paper 150M [29] studied the two most coherent ordinary channels:

$$R^+_{\text{patch}}, \qquad R_{\text{trans}}.$$

Those channels represent coherent aligned-patch support and protected transition-layer support. Paper 150M's central target was:

$$R^+_{\text{patch}} + R_{\text{trans}} \Longrightarrow \text{absorbable cost, finite lifetime, or named exit channel.}$$

The named ordinary exit channels from Paper 150M are precisely the main channels of Paper 150N:

$$R_{\text{frag}}, \qquad R_{\text{scale}}, \qquad R_{\text{low}}.$$

If a coherent aligned patch breaks into many components, the route exits to R_{frag}. If a coherent or transition-layer structure becomes visible only after filtering, the route exits to R_{scale}. If the support slips outside the selected high-vorticity mask or flickers across thresholds, the route exits to R_{low}.

Thus Paper 150N picks up exactly where Paper 150M stops:

$$\text{Paper 150M: coherent structure,}$$

$$\text{Paper 150N: broken, filtered, and threshold-shifting ordinary structure.}$$

The combined ordinary-channel picture is:

$$R^+_{\text{patch}} + R_{\text{trans}} \quad \text{handled by Paper 150M,}$$

and

$$R_{\text{frag}} + R_{\text{scale}} + R_{\text{low}} \quad \text{handled by Paper 150N.}$$

If both layers succeed, then the ordinary-channel remainder satisfies

$$R^+_{\text{patch}} + R_{\text{trans}} + R_{\text{frag}} + R_{\text{scale}} + R_{\text{low}} \le (\delta_{\text{coh}} + \delta_{\text{rem,ord}})D + (C_{\text{coh}} + C_{\text{rem,ord}})E_\omega$$

in pointwise or subinterval-stable integrated form.

The final assembly still requires control or reduction of R_{path} and preservation of the total margin.

11.5 Role of fragmentation after Paper 150M

Fragmentation is the first downstream ordinary channel after coherent-channel failure. If an aligned patch or transition-layer core breaks into many stretching-active pieces, coherent-channel control has not automatically succeeded. The obstruction has changed form.

Paper 150N assigns this route to

$$R_{\text{frag}}.$$

The fragmentation question is whether many components pay interface cost, lose coherent alignment, become short-lived, weaken by separation, become scale-local, enter the complement, or become residual pathology.

Reconnection is included in this downstream question. If fragmented components merge, braid, or form necklace-like structures while preserving positive stretching, the route must pay reconnection-neighborhood gradient cost, return to a coherent channel, become scale-local, enter the complement, or exit to R_{path}. Reconnection is therefore not an untracked loophole in the fragmentation layer.

Thus fragmentation is not a vague failure of Paper 150M. It is a named dependency:

$$R_{\text{patch}}^{+} \text{ or } R_{\text{trans}} \Longrightarrow R_{\text{frag}} \Longrightarrow \text{cost, lower-order behavior, or downstream channel.}$$

This keeps the proof architecture modular.

11.6 Role of scale-locality after Paper 150M

Scale-locality is the second downstream ordinary channel after coherent-channel failure. A coherent structure may become thin, nested, filtered, or visible only at a particular scale. A fragmented structure may also become scale-dependent.

Paper 150N assigns this route to

$$R_{\text{scale}}.$$

The scale-local question is whether the declared scale family captures the positive-stretching contribution and whether that contribution pays high-frequency, transfer, scale-boundary, or finite-lifetime cost.

If a route is not visible because the scale family is too coarse, the result is diagnostic incompleteness, not theorem-level control. If the route remains dangerous across a sufficiently complete scale family, it exits to R_{path}. If the scale family is so dense or redundant that the coefficient sum diverges, scale-local control also fails by budget inflation.

Thus scale-locality protects the proof from treating unresolved diagnostic failure as either success or pathology without declaration, while also forcing the scale-family budget to remain finite.

11.7 Role of complement control after Paper 150M

The complement channel protects the proof from threshold leakage. A coherent patch, transition layer, fragmented support, or scale-local structure may slip outside the selected high-vorticity mask:

$$\Omega_\kappa(t) = \{x : |\omega(x,t)| > \kappa(t)\}.$$

It may also flicker across the threshold or sit in a smooth transition band.

Paper 150N assigns this route to

$$R_{\text{low}}.$$

The complement question is whether the positive stretching outside the high-vorticity mask is lower-order, absorbed by dissipation and enstrophy, controlled by smooth threshold weights, confined to a summable threshold-flicker sequence, controlled by finite-budget flicker accounting, or reassigned to scale-local or pathological channels.

Thus complement control prevents cutoff choices from hiding ordinary positive stretching.

11.8 Relation to residual pathology

Paper 150N allows ordinary channels to exit to

$$R_{\text{path}}.$$

This is not a proof of control. It is a disciplined failure route.

If fragmentation preserves positive stretching while avoiding interface cost, reconnection cost, alignment loss, finite lifetime, scale-local classification, and complement assignment, then the route becomes residual pathology. If scale-local support evades a sufficiently complete scale family without paying high-frequency cost, or if its scale budget diverges, it becomes residual pathology. If complement stretching survives smooth thresholds, threshold-band control, flicker summability, finite-budget flicker control, and lower-order estimates, it becomes residual pathology.

Thus Paper 150N narrows the ordinary-channel problem. It says:

$$R_{\text{frag}},\ R_{\text{scale}},\ R_{\text{low}} \Longrightarrow \text{absorbable, lower-order, finite-budget controlled, reassigned, or } R_{\text{path}}.$$

The pathological-channel layer must then determine whether the residual route reduces to ordinary channels or becomes absorbable:

$$R_{\text{path}} \Longrightarrow \text{ordinary-channel reduction or absorbable residual cost.}$$

Paper 150N does not prove that final step. It makes sure the residual obstruction is named.

11.9 Relation to margin sharpness

The central quantitative constraint is the Paper 150J margin:

$$\theta + \delta_{\text{coh}} + \delta_{\text{rem,ord}} + \delta_{\text{path}} < 1.$$

Paper 150M contributes δ_{coh}. Paper 150N contributes $\delta_{\text{rem,ord}}$. The pathological-channel layer contributes δ_{path}. A later coefficient-recovery paper must show that the sum remains below one with reserve.

This means Paper 150N can fail even if every remaining ordinary channel is estimated. If

$$\theta + \delta_{\text{coh}} + \delta_{\text{rem,ord}} + \delta_{\text{path}} \geq 1,$$

then the estimates are too expensive to close the enstrophy inequality.

A sharper practical target is that Paper 150N should leave a positive reserve:

$$\delta_{\mathrm{rem,ord}} \leq (1 - \theta - \delta_{\mathrm{coh}}) - \delta_{\mathrm{reserve,N}},$$

where

$$\delta_{\mathrm{reserve,N}} > 0$$

is reserved for residual pathology and final coefficient recovery.

Thus Paper 150N is also a budget paper. Its estimates must be not merely true, but sharp enough.

11.10 Future bridge sequence after Paper 150N

After Paper 150N, the ordinary-channel layer has been split into two parts:

$$\text{Paper 150M: } R^{+}_{\mathrm{patch}} + R_{\mathrm{trans}},$$

and

$$\text{Paper 150N: } R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}}.$$

The natural next steps are:

Paper 150O: residual pathological-channel refinement after ordinary-channel control,

Paper 150P: margin sharpness and coefficient recovery,

Paper 150Q: final bridge assembly after visibility, ordinary-channel control, residual-pathology refinement, and m

Paper 150O should not duplicate the earlier pathological-channel framework. Its role should be to revisit the residual R_{path} routes that remain after Papers 150M and 150N have reduced or controlled the ordinary channels.

Paper 150P should then focus on the quantitative issue:

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1.$$

Even a complete classification is not enough unless the coefficients preserve the dissipation reserve.

Paper 150Q can then assemble the refined bridge stack.

11.11 Summary

Paper 150N fits into the 150-series as the remaining ordinary-channel control paper. Paper 150L [28] supplies visibility. Paper 150M [29] controls coherent ordinary channels. Paper 150N controls the broken, filtered, and threshold-shifting ordinary channels. Paper 150K [27] supplies the accounting rules that make channel estimates usable. Paper 150J [26] supplies the conditional assembly theorem.

The role of Paper 150N is therefore:

$R_{\text{frag}}+R_{\text{scale}}+R_{\text{low}} \Longrightarrow$ absorbable cost, lower-order behavior, finite-budget control, reassignment, or R_{path}.

If the favorable estimates hold, Paper 150N supplies

$$R_{\text{rem,ord}} \le \delta_{\text{rem,ord}} D + C_{\text{rem,ord}} E_\omega$$

in pointwise or subinterval-stable integrated form. If they fail, the failure is named: fragmentation failure, reconnection failure, scale-local failure, complement failure, double-counting, margin exhaustion, hidden spike, or residual pathology.

This keeps the proof program modular and prevents ordinary-channel failure from becoming an undefined remainder.

11.12 Toward runaway-exclusion functionals

The remaining ordinary-channel problem can also be viewed as a runaway-exclusion problem for selected time-dependent functionals. For each channel R_j, one seeks a quantity $Y_j(t)$ measuring the dangerous positive-stretching reservoir assigned to that channel and then derives an inequality of the schematic form

$$\frac{\mathrm{d}Y_j}{\mathrm{d}t} \le \mathcal{G}_j(t)Y_j(t) - \mathcal{C}_j(t) + \mathcal{L}_j(t),$$

where $\mathcal{G}_j(t)$ is the local amplification rate, $\mathcal{C}_j(t)$ is the channel cost, and $\mathcal{L}_j(t)$ is lower-order. A runaway channel requires persistent positive growth while avoiding the associated cost. The purpose of the bridge estimates is to show that such avoidance forces finite lifetime, channel transition, lower-order behavior, or residual pathology.

For the fragmentation channel, a natural functional is the fragmented positive-stretching reservoir,

$$Y_{\text{frag}}(t) = \sum_j \int_{A_j(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V.$$

A schematic differential target is

$$\frac{\mathrm{d}Y_{\text{frag}}}{\mathrm{d}t} \le \mathcal{G}_{\text{frag}}(t)Y_{\text{frag}}(t) - \mathcal{C}_{\text{int}}(t) - \mathcal{C}_{\text{rec}}(t) - \mathcal{C}_{\text{disp}}(t) + \mathcal{L}_{\text{frag}}(t),$$

where the negative terms represent interface cost, reconnection cost, and component-dispersion or alignment-loss cost.

For the scale-local channel, one may use

$$Y_{\text{scale}}(t) = \sum_{\ell\in\mathcal{L}} \int_\Omega \chi_\ell |\omega_\ell|^2 a_\ell^+\ \mathrm{d}V,$$

with the target that growth is offset by high-frequency cost, scale-transfer cost, finite scale-budget accounting, or lower-order behavior.

For the complement channel, one may use

$$Y_{\text{low}}(t) = \int_\Omega (1 - W_\kappa(|\omega|))|\omega|^2 a^+\ \mathrm{d}V,$$

with the target that growth is offset by lower-order enstrophy control, smooth-threshold leakage, threshold-band cost, or finite-budget flicker accounting.

Paper 150N does not prove these differential inequalities. It identifies the channel functionals and the cost terms that such a calculus must control. A later coefficient-recovery paper can use these functionals to turn the channel taxonomy into explicit runaway-exclusion estimates.

12 Conclusion

Paper 150N continued the ordinary-channel control layer of the 150-series high-vorticity pinching program for the three-dimensional incompressible Navier–Stokes equations. Paper 150M [29] treated the two most coherent ordinary channels,

$$R^{+}_{\mathrm{patch}} \qquad \text{and} \qquad R_{\mathrm{trans}}.$$

The present paper studied the remaining ordinary channels:

$$R_{\mathrm{frag}}, \qquad R_{\mathrm{scale}}, \qquad R_{\mathrm{low}}.$$

These channels represent what happens after coherent structure fails to remain a single aligned patch or protected transition-layer core. A coherent route may break into many components, reconnect, become visible only after filtering, or slip across the chosen high-vorticity threshold. Paper 150N organized these broken, filtered, reconnecting, and threshold-shifting routes into explicit control targets.

The starting point was the classical enstrophy balance

$$\frac{\mathrm{d}E_\omega}{\mathrm{d}t} = P(t) - D(t),$$

where

$$P(t) = \int_\Omega \omega_i S_{ij} \omega_j \ \mathrm{d}V$$

is vortex stretching and

$$D(t) = \nu \int_\Omega |\nabla \omega|^2 \ \mathrm{d}V$$

is viscous enstrophy dissipation. The dangerous contribution is the positive stretching density

$$|\omega|^2 a^+(x,t), \qquad a^+(x,t) = \max\{n_i S_{ij} n_j, 0\}.$$

The available geometric cost is organized by

$$|\nabla \omega|^2 = |\nabla |\omega||^2 + |\omega|^2 |\nabla n|^2.$$

The first channel studied was fragmentation:

$$R_{\mathrm{frag}}.$$

Fragmentation is activated when positive stretching is distributed across many separated, semi-separated, moving, reconnecting, or intermittently active components. The paper defined component positive-stretching fractions

$$\pi_j^+(t) = \frac{P_j^+(t)}{\sum_k P_k^+(t) + \varepsilon}$$

and the effective positive component count

$$N_{\mathrm{eff}}^+(t) = \left(\sum_j (\pi_j^+(t))^2 \right)^{-1}.$$

Fragmentation is favorable only if it creates interface cost, reconnection cost, boundary cost, separation cost, alignment loss, finite component lifetime, lower-order behavior, or a named channel exit. It is dangerous if many components collectively preserve positive stretching while paying too little cost.

The fragmentation target was

$$R_{\mathrm{frag}}(t) \le \delta_{\mathrm{frag}} D(t) + C_{\mathrm{frag}} E_\omega(t),$$

or the subinterval-stable integrated analogue. If direct absorbability fails, the fragmented route must lose alignment, become short-lived, weaken by separation, pay reconnection cost, exit to R_{scale}, exit to R_{low}, or become residual pathology R_{path}.

The second channel studied was scale-local visibility:

$$R_{\mathrm{scale}}.$$

This channel is activated when dangerous positive stretching is not adequately visible at the full-field level but appears after filtering or scale decomposition. The paper introduced a declared scale family

$$\mathcal{L} = \{\ell_1, \ell_2, \ldots, \ell_N\},$$

with filtered fields

$$u_\ell = G_\ell * u, \qquad \omega_\ell = \nabla \times u_\ell, \qquad S_\ell = \frac{1}{2}(\nabla u_\ell + \nabla u_\ell^T).$$

Scale-local positive stretching was represented by

$$|\omega_\ell|^2 a_\ell^+(x,t).$$

The scale-local target was

$$R_{\mathrm{scale}}(t) \le \delta_{\mathrm{scale}} D(t) + C_{\mathrm{scale}} E_\omega(t),$$

or the corresponding subinterval-stable integrated estimate. The paper emphasized that scale-local control requires a declared scale family with controlled overlap and finite scale budget. If the smallest declared scale is too coarse, then the result is diagnostic incompleteness, not theorem-level control. If a route remains dangerous across a sufficiently complete declared scale family while avoiding gradient cost, transfer cost, complement assignment, lower-order behavior, and finite scale-budget accounting, then it exits to residual pathology.

The third channel studied was low-vorticity complement stretching:

$$R_{\text{low}}.$$

This channel is activated when positive stretching lies outside the selected high-vorticity mask, near a threshold boundary, or in the complement of a smooth high-vorticity weight. For a sharp threshold

$$\Omega_\kappa(t) = \{x \in \Omega : |\omega(x,t)| > \kappa(t)\},$$

the complement contribution is schematically

$$\int_{\Omega \setminus \Omega_\kappa(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V.$$

For a smooth high-vorticity weight $W_\kappa(|\omega|)$, the exact split

$$W_\kappa(|\omega|) + \big(1 - W_\kappa(|\omega|)\big) = 1$$

prevents positive stretching from disappearing through a cutoff.

The complement target was

$$R_{\text{low}}(t) \le \delta_{\text{low}} D(t) + C_{\text{low}} E_\omega(t),$$

or a subinterval-stable integrated analogue. Complement control may occur through lower-order enstrophy control, smooth-threshold stabilization, threshold-band control, threshold-flicker summability, finite-budget flicker control, broad-support estimates, or threshold-motion control. If these fail, the route exits to R_{scale} or R_{path}.

The combined remaining ordinary-channel contribution was defined by

$$R_{\text{rem,ord}}(t) = R_{\text{frag}}(t) + R_{\text{scale}}(t) + R_{\text{low}}(t).$$

The desired combined estimate is

$$R_{\text{rem,ord}}(t) \le \delta_{\text{rem,ord}} D(t) + C_{\text{rem,ord}} E_\omega(t),$$

with

$$\delta_{\text{rem,ord}} = \delta_{\text{frag}} + \delta_{\text{scale}} + \delta_{\text{low}}, \qquad C_{\text{rem,ord}} = C_{\text{frag}} + C_{\text{scale}} + C_{\text{low}}.$$

For moving, intermittent, reconnecting, scale-dependent, or threshold-flickering routes, the corresponding integrated estimate is

$$\int_{t_0}^{t} R_{\text{rem,ord}}(s)\ \mathrm{d}s \le \delta_{\text{rem,ord}} \int_{t_0}^{t} D(s)\ \mathrm{d}s + C_{\text{rem,ord}} \int_{t_0}^{t} E_\omega(s)\ \mathrm{d}s + C_{0,\text{rem,ord}}$$

for every $t \in I$.

The subinterval condition is essential. A full-interval integrated estimate may hide a transient enstrophy spike. Paper 150N therefore follows the Paper 150K [27] accounting rule: integrated channel estimates must hold on every subinterval or through stopping-time partial sums that control every intermediate time. Burst constants must also be summable; otherwise fragmentation bursts, reconnection events, scale-transfer bursts, or threshold-flicker intervals could accumulate into an uncontrolled contribution.

The main theorem of Paper 150N was conditional. It stated that if fragmentation, scale-local visibility, and complement stretching satisfy their respective control alternatives, if channel transitions avoid double-counting, if integrated estimates are subinterval-stable, if scale and event budgets are finite or summable, and if the margin remains positive, then

$$R_{\mathrm{frag}}, \qquad R_{\mathrm{scale}}, \qquad R_{\mathrm{low}}$$

do not remain independent ordinary-channel obstructions. Each active route is either absorbed, made lower-order, paid for by ordinary cost, controlled by finite scale or event accounting, reassigned to another ordinary channel, or moved into residual pathology.

The relevant Paper 150J [26] margin is

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1.$$

Here θ is the primary depletion coefficient, δ_{coh} is the coherent-channel coefficient from Paper 150M, $\delta_{\mathrm{rem,ord}}$ is the remaining ordinary-channel coefficient from Paper 150N, and δ_{path} is the residual pathological-channel coefficient. This margin is the quantitative budget line of the proof. It is not enough to identify or estimate the channels. The estimates must be sharp enough to leave dissipation reserve.

The main review burden for this paper is concentrated in four places. First, reconnection events must not be allowed to preserve positive stretching without interface or gradient cost. Second, the declared scale family must be dense enough to detect active scale-local structure while sparse or orthogonal enough to keep the scale budget finite. Third, threshold flicker must be controlled by finite crossing, summable constants, or finite-budget accounting. Fourth, the coefficients δ_{frag}, δ_{scale}, and δ_{low} must be small enough that the ordinary-channel layer leaves reserve for R_{path}. These are not cosmetic concerns. They are the main theorem targets left after Paper 150N.

The paper also made the failure modes explicit. Fragmentation can fail if many components preserve positive stretching without interface cost, reconnection cost, alignment loss, finite lifetime, or summable burst accounting. Scale-local control can fail if filtered positive stretching persists without high-frequency cost, if the declared scale family is incomplete, if the scale budget diverges, if scale transfer has nonsummable constants, or if classification changes under nearby filters. Complement control can fail if stretching outside the high-vorticity mask is not lower-order, if threshold-band stretching remains significant, if threshold flicker is nonsummable, if finite-budget flicker control is absent, or if smooth threshold weights do not stabilize the split.

The global failure modes are double-counting, margin exhaustion, hidden integrated spikes, downstream pathological overload, and numerical overinterpretation. These are not vague objections. They are named obstructions that later work must address.

Paper 150N therefore advances the 150-series by completing the ordinary-channel control map at the theorem-target level. Paper 150M handled coherent structure:

$$R^{+}_{\mathrm{patch}} + R_{\mathrm{trans}}.$$

Paper 150N handled broken, filtered, reconnecting, and threshold-shifting ordinary structure:

$$R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}}.$$

Together, these papers reduce the ordinary-channel burden to explicit absorbability estimates, lower-order routes, finite-budget accounting, or named exits to residual pathology.

The remaining work is now clearer. The next bridge layer should refine the residual pathological channel after ordinary-channel control, focusing on any R_{path} routes left by Papers 150M and 150N. After that, the program must recover sharp coefficients and verify that the total dissipation margin remains positive:

$$\theta + \delta_{\text{coh}} + \delta_{\text{rem,ord}} + \delta_{\text{path}} < 1.$$

Only then can the final bridge assembly be tested as a regularity pathway.

In summary, Paper 150N does not prove unconditional Navier–Stokes regularity. It supplies the remaining ordinary-channel bridge target. If fragmentation, scale-local visibility, and complement stretching are absorbed, made lower-order, finite-budget controlled, or assigned to residual pathology without double-counting and with margin preserved, then the ordinary-channel layer no longer remains an undefined obstruction. The proof program can then move to residual pathology, coefficient sharpness, and final assembly.

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A Glossary of Terms and Symbols

A.1 Key Terms

Navier–Stokes equations The classical equations for viscous incompressible fluid motion:

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \qquad \nabla \cdot u = 0.$$

Velocity field The vector field $u(x,t)$ giving the local velocity of the fluid.

Pressure The scalar field $p(x,t)$ enforcing incompressibility. In the vorticity formulation, pressure is not explicit, but its nonlocal effects remain through the velocity gradient and strain tensor.

Incompressibility The condition

$$\nabla \cdot u = 0.$$

It means the fluid does not locally expand or compress.

Vorticity The local rotational part of the flow:

$$\omega = \nabla \times u.$$

Vorticity magnitude The scalar strength of local rotation:

$$|\omega|.$$

Vorticity direction The unit vector

$$n = \frac{\omega}{|\omega|}$$

where $|\omega| > 0$. It records the direction of local rotation.

Strain tensor The symmetric part of the velocity gradient:

$$S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i).$$

It measures local stretching and compression.

Vortex stretching The nonlinear three-dimensional amplification mechanism:

$$P(t) = \int_\Omega \omega_i S_{ij} \omega_j \, \mathrm{d}V.$$

It is the term that can increase enstrophy.

Enstrophy The total squared vorticity:

$$E_\omega(t) = \frac{1}{2} \int_\Omega |\omega|^2 \, \mathrm{d}V.$$

Enstrophy dissipation The viscous damping term:

$$D(t) = \nu \int_\Omega |\nabla \omega|^2 \, \mathrm{d}V.$$

Enstrophy balance The identity

$$\frac{\mathrm{d}E_\omega}{\mathrm{d}t} = P - D.$$

It says enstrophy grows when vortex stretching exceeds dissipation and decays when dissipation exceeds stretching.

Positive stretching The part of vortex stretching that locally increases enstrophy:

$$|\omega|^2 a^+(x,t), \qquad a^+(x,t) = \max\{n_i S_{ij} n_j, 0\}.$$

Positive stretching density The nonnegative density

$$|\omega|^2 a^+(x,t).$$

This is the main stretching reservoir assigned to the channels studied in this paper.

Positive stretching reservoir The total positive stretching:

$$P^+(t) = \int_\Omega |\omega|^2 a^+(x,t) \, \mathrm{d}V.$$

Integrated positive stretching Positive stretching accumulated over a time interval:

$$\int_I \int_\Omega |\omega|^2 a^+(x,t) \, \mathrm{d}V \, \mathrm{d}t.$$

This is important for moving, intermittent, fragmented, scale-local, and threshold-flickering routes.

High-vorticity pinching program The 150-series approach that studies whether dangerous high-vorticity amplification must pay geometric cost, lose alignment, leak, fragment, become scale-local, enter the complement, or become residual pathology.

Primary depletion regime The main high-vorticity pinching/depletion architecture, represented schematically by

$$P(t) \le \theta D(t) + C E_\omega(t) + R_\kappa(t), \qquad 0 \le \theta < 1.$$

Primary depletion coefficient The coefficient θ in the primary depleted stretching estimate. It measures how much dissipation is consumed before channel remainders are added.

Remainder The part of stretching not controlled by the primary depleted estimate:

$$R_\kappa.$$

Channel decomposition The splitting of the remainder into named channels:

$$R_\kappa = R^+_{\mathrm{patch}} + R_{\mathrm{trans}} + R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}} + R_{\mathrm{path}}.$$

Ordinary channels The structured non-pathological channels:

$$R^+_{\mathrm{patch}}, \quad R_{\mathrm{trans}}, \quad R_{\mathrm{frag}}, \quad R_{\mathrm{scale}}, \quad R_{\mathrm{low}}.$$

Coherent channels The two ordinary channels studied in Paper 150M:

$$R^+_{\mathrm{patch}}, \qquad R_{\mathrm{trans}}.$$

Remaining ordinary channels The three ordinary channels studied in Paper 150N:

$$R_{\mathrm{frag}}, \qquad R_{\mathrm{scale}}, \qquad R_{\mathrm{low}}.$$

Fragmentation channel The channel

$$R_{\mathrm{frag}}$$

in which positive stretching is distributed across many separated, semi-separated, moving, reconnecting, or intermittently active components.

Fragmented support A stretching-active support of the form

$$A(t) = \bigcup_j A_j(t),$$

where the pieces $A_j(t)$ carry portions of positive stretching.

Component positive stretching The positive stretching carried by one component:

$$P_j^+(t) = \int_{A_j(t)} |\omega|^2 a^+(x,t)\ \mathrm{d}V.$$

Positive-stretching fraction The fraction of fragmented positive stretching carried by component j:

$$\pi_j^+(t) = \frac{P_j^+(t)}{\sum_k P_k^+(t) + \varepsilon}.$$

Effective positive component count A diagnostic measuring how many components effectively carry positive stretching:

$$N^+_{\mathrm{eff}}(t) = \left(\sum_j (\pi_j^+(t))^2 \right)^{-1}.$$

Interface cost Gradient cost generated near boundaries between fragmented components:

$$C_{\mathrm{int}}(t) = \nu \sum_j \int_{N_r(\partial A_j(t))} |\nabla\omega|^2 \, \mathrm{d}V.$$

Reconnection event A merge, split, braid, or support-exchange event among fragmented components. In Paper 150N, reconnection must appear as interface cost, coherent reclassification, scale-local structure, complement behavior, or residual pathology.

Reconnection cost The enstrophy-gradient cost associated with a reconnection neighborhood:

$$C_{\mathrm{rec}}(t) = \nu \int_{N_r(\mathcal{R}_{\mathrm{rec}}(t))} |\nabla\omega|^2 \, \mathrm{d}V.$$

Alignment loss A route by which a component, patch, or scale-local support stops being dangerous because its positive strain alignment decreases.

Component lifetime The time during which a fragmented component remains trackable and stretching-active.

Fragmentation burst A short interval during which fragmented positive stretching becomes significant.

Scale-local channel The channel

$$R_{\mathrm{scale}}$$

in which dangerous positive stretching becomes visible only after filtering or scale decomposition.

Declared scale family A declared set of scales

$$\mathcal{L} = \{\ell_1, \ell_2, \ldots, \ell_N\}$$

used to test scale-local visibility.

Scale-family density The requirement that the declared scale family be fine enough to detect active scale-local support while keeping overlap and coefficient sums finite.

Resolution floor The smallest declared filter or diagnostic scale:

$$\ell_{\min} = \min_{\ell\in\mathcal{L}} \ell.$$

If the active support lies below this floor, the result is diagnostic incompleteness unless unresolved-scale cost or residual pathology is proved.

Finite scale budget The requirement that the sum or integral of scale-local coefficients remain finite:

$$\sum_{\ell\in\mathcal{L}} \delta_\ell < \infty$$

or

$$\int_{\mathcal{L}} \delta(\ell) \, \mathrm{d}\mu(\ell) < \infty.$$

Filter A smoothing or localization operator G_ℓ at scale ℓ. The filtered fields are

$$u_\ell = G_\ell * u, \qquad \omega_\ell = \nabla \times u_\ell.$$

Scale-local strain tensor The strain tensor of the filtered velocity field:

$$S_\ell = \frac{1}{2}(\nabla u_\ell + \nabla u_\ell^T).$$

Scale-local alignment factor The alignment factor at scale ℓ:

$$a_\ell(x,t) = n_{\ell,i} S_{\ell,ij} n_{\ell,j}.$$

Scale-local positive stretching The filtered positive stretching density:

$$|\omega_\ell|^2 a_\ell^+(x,t), \qquad a_\ell^+(x,t) = \max\{a_\ell(x,t), 0\}.$$

Resolved scale-local visibility A route is resolved when a declared scale $\ell \in \mathcal{L}$ detects significant organized positive stretching.

Unresolved diagnostic failure A failure caused by a scale family that is too coarse to see the active support. This is not theorem-level control.

Scale evasion A residual failure in which dangerous positive stretching remains active across a sufficiently complete declared scale family while avoiding scale-local control.

High-frequency cost Gradient or dissipation cost associated with small-scale or high-frequency structure.

Scale-transfer cost Cost associated with positive stretching moving between scales.

Scale-boundary cost Gradient or interface cost generated near the boundary of a scale-local support.

Low-vorticity complement channel The channel

$$R_{\text{low}}$$

in which positive stretching lies outside the selected high-vorticity region, near a threshold boundary, or in the complement of a smooth high-vorticity weight.

High-vorticity region For a threshold $\kappa(t)$, the high-vorticity region is

$$\Omega_\kappa(t) = \{x \in \Omega : |\omega(x,t)| > \kappa(t)\}.$$

Complement region The region outside the selected high-vorticity mask:

$$\Omega \setminus \Omega_\kappa(t).$$

Smooth high-vorticity weight A smooth function $W_\kappa(|\omega|) \in [0,1]$ used to replace a sharp threshold.

Smooth complement weight The complement of a smooth high-vorticity weight:

$$1 - W_\kappa(|\omega|).$$

Threshold band A band near the high-vorticity boundary:

$$B_\kappa(t) = \{x \in \Omega : \kappa_-(t) \le |\omega(x,t)| \le \kappa_+(t)\}.$$

Threshold flicker Repeated movement of a stretching-active support across a high-vorticity threshold or smooth threshold band.

Zeno-style threshold flicker Infinitely many threshold crossings in finite time. Such behavior is controlled only if the associated constants are summable.

Finite-budget flicker control A threshold-flicker control mechanism in which each significant crossing consumes part of a finite budget, implying finitely many significant crossings or summable residual constants.

Broad low-intensity complement support A large region outside the high-vorticity mask carrying weak but cumulatively significant positive stretching.

Threshold motion Time variation of the threshold $\kappa(t)$, which can move the high-vorticity region even if the underlying flow changes smoothly.

Residual pathology A route that remains dangerous after ordinary-channel control, lower-order reduction, scale-local classification, complement assignment, and absorbability fail.

Pathological concentration channel The residual channel

$$R_{\text{path}}$$

assigned to dangerous concentration after ordinary-channel tests fail.

Remaining ordinary-channel contribution The combined channel contribution studied in Paper 150N:

$$R_{\text{rem,ord}} = R_{\text{frag}} + R_{\text{scale}} + R_{\text{low}}.$$

Remaining ordinary-channel coefficient The combined dissipation coefficient

$$\delta_{\text{rem,ord}} = \delta_{\text{frag}} + \delta_{\text{scale}} + \delta_{\text{low}}.$$

Coherent-channel coefficient The coefficient from Paper 150M:

$$\delta_{\text{coh}} = \delta_{\text{patch}} + \delta_{\text{trans}}.$$

Margin preservation The requirement that the total dissipation coefficient remain below one:

$$\theta + \delta_{\text{coh}} + \delta_{\text{rem,ord}} + \delta_{\text{path}} < 1.$$

Margin exhaustion A failure mode in which estimates exist but consume all available dissipation:

$$\theta + \delta_{\text{coh}} + \delta_{\text{rem,ord}} + \delta_{\text{path}} \ge 1.$$

Budget hierarchy The intended ordering of remaining ordinary-channel costs: fragmentation should mainly pay interface or separation cost, scale-locality should mainly pay high-frequency or transfer cost, and complement stretching should mainly be lower-order or threshold-accounting dominated.

No double-counting The requirement that the same positive-stretching contribution not be charged simultaneously to multiple channels unless an explicit overlap budget is included.

Partition of unity A family of weights $\chi_j \geq 0$ satisfying

$$\sum_j \chi_j = 1.$$

It assigns positive stretching exactly once.

Bounded overlap A controlled-overlap condition

$$\sum_j \chi_j \leq K_{\mathrm{ov}}.$$

If $K_{\mathrm{ov}} > 1$, the overlap must be charged to the margin.

Subinterval-stable integrated estimate An estimate that holds on every subinterval $[t_0, t] \subset I$. This is required to avoid hidden spikes.

Hidden spike A failure mode in which a full-interval integrated estimate appears controlled while a large intermediate enstrophy spike occurs.

Burst summability The requirement that constants from burst or stopping-time estimates satisfy

$$\sum_j C_{0,j} < \infty.$$

Conditional remaining-ordinary-channel control theorem The Paper 150N theorem stating that, under the fragmentation, scale-local, complement, no-double-counting, subinterval-stability, scale-density, flicker-summability, and margin hypotheses,

$$R_{\mathrm{frag}} + R_{\mathrm{scale}} + R_{\mathrm{low}}$$

is absorbable, lower-order, finite-budget controlled, reassigned, or residual pathological.

A.2 Symbols

Table 1: Symbols used in Paper 150N.

Symbol	Pronunciation	Meaning
u	"u"	Velocity field
p	"p"	Pressure field
ν	"nu"	Kinematic viscosity

Symbol	Pronunciation	Meaning
Ω	“Omega”	Spatial domain
$\mathbb{T}^3$	“three-torus”	Periodic three-dimensional domain
t	“t”	Time
∇	“gradient” or “del”	Spatial derivative
Δ	“Laplacian”	Diffusion operator
ω	“omega”	Vorticity field
$\lvert\omega\rvert$	“omega magnitude”	Vorticity strength
n	“n”	Unit vorticity direction
S_{ij}	“S i j”	Strain tensor
a	“a”	Alignment factor $n_i S_{ij} n_j$
a^+	“a plus”	Positive part of the alignment factor
E_ω	“E omega”	Enstrophy
P	“P”	Vortex stretching
P^+	“P plus”	Positive stretching reservoir
D	“D”	Enstrophy dissipation
R_κ	“R kappa”	Full channel remainder
R^+_{patch}	“R patch plus”	Coherent aligned-patch channel
R_{trans}	“R transition”	Transition-layer channel
R_{frag}	“R fragmentation”	Fragmentation channel
R_{scale}	“R scale”	Scale-local channel
R_{low}	“R low”	Low-vorticity complement channel
R_{path}	“R path”	Pathological concentration channel
R_{rec}	“R reconnection”	Positive-stretching contribution carried by reconnection regions
R_{coh}	“R coherent”	Coherent-channel contribution $R^+_{\text{patch}} + R_{\text{trans}}$
$R_{\text{rem,ord}}$	“R remaining ordinary”	Remaining ordinary-channel contribution $R_{\text{frag}} + R_{\text{scale}} + R_{\text{low}}$
Ω_κ	“Omega kappa”	High-vorticity region above threshold κ
κ	“kappa”	High-vorticity threshold
W_κ	“W kappa”	Smooth high-vorticity weight
$1 - W_\kappa$	“one minus W kappa”	Smooth complement weight
B_κ	“B kappa”	Threshold band
$A(t)$	“A of t”	Fragmented stretching-active support
$A_j(t)$	“A j of t”	Component of fragmented support
P^+_j	“P j plus”	Positive stretching carried by component A_j
π^+_j	“pi j plus”	Positive-stretching fraction carried by component j
N^+_{eff}	“N effective plus”	Effective number of positive-stretching components
C_{int}	“C interface”	Fragmentation interface cost
C_{rec}	“C reconnection”	Reconnection-neighborhood gradient cost

Symbol	Pronunciation	Meaning
δ_{rec}	"delta reconnection"	Dissipation coefficient for reconnection cost
$\mathcal{L}$	"script L"	Declared scale family
ℓ	"ell"	Filter scale
$\ell_{\min}$	"ell min"	Smallest declared scale in the scale family
N_{scale}	"N scale"	Number of scales in the declared finite scale family
G_ℓ	"G ell"	Filter at scale ℓ
u_ℓ	"u ell"	Filtered velocity field
ω_ℓ	"omega ell"	Filtered vorticity field
S_ℓ	"S ell"	Filtered strain tensor
n_ℓ	"n ell"	Filtered vorticity direction
a_ℓ	"a ell"	Filtered alignment factor
a_ℓ^+	"a ell plus"	Positive part of filtered alignment factor
P_ℓ^+	"P ell plus"	Scale-local positive-stretching reservoir
χ_ℓ	"chi ell"	Scale-local weight
D_{hf}	"D high frequency"	High-frequency or small-scale dissipation contribution
D_{scale}	"D scale"	Dissipation, gradient, transfer, or high-frequency cost associated with active scale-local structure
$T_{\ell\to\ell'}$	"T ell to ell prime"	Schematic transfer from scale ℓ to scale ℓ'
θ	"theta"	Primary depletion coefficient
δ_{coh}	"delta coherent"	Coherent-channel coefficient from Paper 150M
δ_{frag}	"delta fragmentation"	Fragmentation dissipation coefficient
δ_{scale}	"delta scale"	Scale-local dissipation coefficient
$\delta_{\mathrm{scale,eff}}$	"delta scale effective"	Scale-local coefficient after overlap or scale-budget adjustment
δ_{low}	"delta low"	Complement dissipation coefficient
$\delta_{\mathrm{rem,ord}}$	"delta remaining ordinary"	Combined coefficient $\delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}}$
δ_{path}	"delta path"	Pathological-channel coefficient
$\delta_{\mathrm{reserve,N}}$	"delta reserve N"	Reserve left after Paper 150N for residual pathology and coefficient recovery
$\delta_{\mathrm{reserve,frag}}$	"delta reserve fragmentation"	Reserve left after fragmentation control for downstream channels
$\delta_{\mathrm{reserve,scale}}$	"delta reserve scale"	Reserve left after scale-local control for downstream channels
$\delta_{\mathrm{reserve,low}}$	"delta reserve low"	Reserve left after complement control for downstream channels
C_{frag}	"C fragmentation"	Lower-order enstrophy coefficient for fragmentation
C_{scale}	"C scale"	Lower-order enstrophy coefficient for scale-local control

Symbol	Pronunciation	Meaning
C_{low}	"C low"	Lower-order enstrophy coefficient for complement control
$C_{\text{rem,ord}}$	"C remaining ordinary"	Combined lower-order coefficient for remaining ordinary channels
C_0	"C zero"	Integrated residual constant
$C_{0,j}$	"C zero j"	Burst or stopping-time residual constant
I	"I"	Time interval
I_j	"I j"	Burst, component, or stopping-time interval
τ_j	"tau j"	Stopping-time endpoint
K_{ov}	"K overlap"	Bounded-overlap constant
B_{flicker}	"B flicker"	Budget density controlling threshold-flicker events
R_{motion}	"R motion"	Threshold-motion contribution
ε	"epsilon"	Small regularization constant
δ_{reserve}	"delta reserve"	Dissipation reserve kept for downstream channels

B Plain-Language Summary

B.1 Plain-Language Summary

This paper is about what happens when dangerous spinning fluid structures stop being clean, coherent objects.

The Navier–Stokes equations describe how fluids move. In three dimensions, spinning parts of a fluid can stretch. Stretching can make the spin stronger. The hard question is whether this strengthening can ever run away without bound.

Earlier papers in the 150-series studied the first visible dangerous structures. Paper 150M focused on coherent structures: aligned patches and transition layers. Those are organized regions where spinning fluid stays lined up with stretching.

Paper 150N studies what happens after those coherent structures break down. There are three main possibilities.

First, the structure can fragment. This means one organized spinning region breaks into many smaller pieces. Fragmentation might help because many pieces create boundaries, interfaces, and misalignment. But it might still be dangerous if many pieces keep stretching together.

Second, the danger can become scale-local. This means the structure is not obvious at the full-field level, but appears after filtering or looking at the right scale. A dangerous structure might hide because we are looking too coarsely. Paper 150N says the scale family must be declared clearly. If the chosen scales are too coarse, that is not proof of control.

Third, the danger can slip through a threshold. High-vorticity arguments often focus on regions where spin is above a cutoff. But positive stretching below the cutoff still matters. The low-vorticity complement channel makes sure that stretching outside the high-vorticity mask is still counted.

The paper asks whether these three channels can be controlled:

$$R_{\text{frag}}, \qquad R_{\text{scale}}, \qquad R_{\text{low}}.$$

The target is to show that their combined contribution is bounded by dissipation and lower-order enstrophy:

$$R_{\text{frag}} + R_{\text{scale}} + R_{\text{low}} \le \delta_{\text{rem,ord}} D + C_{\text{rem,ord}} E_{\omega}.$$

If the estimate is integrated over time, it must hold on every subinterval, so it cannot hide a temporary spike.

The paper does not prove full Navier–Stokes regularity. It gives the next bridge target. It says that broken, filtered, and threshold-shifting structures must either pay cost, become lower-order, or become residual pathology.

B.2 Intuitive Analogy: A Rope Breaking Into Strands

Imagine pulling on a spinning rope.

If the rope stays whole and aligned with the pull, it can stretch efficiently. That was the coherent-channel problem in Paper 150M.

Now imagine the rope frays into many strands. That is fragmentation. Fraying might help because the strands rub, separate, tangle, and lose alignment. But if all the strands keep pulling together, the danger remains.

Paper 150N asks whether the frayed strands must pay a cost.

B.3 Intuitive Analogy: Looking at a Picture With the Wrong Zoom

Imagine looking at a blurry picture.

At one zoom level, you may not see the important structure. When you zoom in, the structure appears. That is scale-local visibility.

A fluid structure may not look dangerous at the full-field level. But after filtering at the right scale, the danger may appear.

Paper 150N says: declare the zoom levels. If the danger is visible at one of them, count it. If the zoom is too coarse, do not pretend the danger is gone.

B.4 Intuitive Analogy: A Fence With Gaps

Imagine building a fence around the most dangerous part of a fire.

If sparks fly outside the fence, they still matter. You cannot say the fire is controlled just because the sparks crossed the boundary.

The high-vorticity threshold is like the fence. The complement channel tracks what happens outside it.

Paper 150N asks whether stretching that slips outside the high-vorticity mask is harmless, controlled, or still dangerous.

B.5 Intuitive Analogy: A Crowd Splitting Into Groups

Imagine a crowd marching in one direction.

If the crowd stays together, it moves powerfully. If it splits into many groups, the motion may weaken. Some groups turn, slow down, or lose coordination.

That is fragmentation. But if all the groups keep marching in the same direction, the crowd can still be powerful.

Paper 150N asks whether fragmented fluid structures lose coordination or keep stretching together.

B.6 Intuitive Analogy: A Flashlight in Fog

Imagine shining a flashlight through fog.

From far away, you may see only a blur. Up close, you may see beams, layers, and streaks.

Scale-local analysis is like changing how you look through the fog. A dangerous structure may only appear after the right filter is used.

Paper 150N says the filters must be chosen honestly and the same structure must not be counted twice.

B.7 Intuitive Analogy: A Speed Limit Line

Imagine classifying cars as fast or slow using a speed limit.

A car going just below the speed limit can still be important. A car crossing above and below the limit repeatedly should not disappear from the traffic count.

The high-vorticity threshold works the same way. Stretching just below the threshold or flickering across it must still be counted.

That is the purpose of R_{low}.

B.8 Intuitive Analogy: A Leaky Accounting Budget

Imagine a project budget.

You can afford repairs only if the total cost stays below the budget. Even if each repair is reasonable, the whole project fails if the total cost is too high.

The Navier–Stokes proof has a dissipation budget:

$$\theta + \delta_{\mathrm{coh}} + \delta_{\mathrm{rem,ord}} + \delta_{\mathrm{path}} < 1.$$

Paper 150N controls the remaining ordinary part:

$$\delta_{\mathrm{rem,ord}} = \delta_{\mathrm{frag}} + \delta_{\mathrm{scale}} + \delta_{\mathrm{low}}.$$

The estimates must leave room for residual pathology.

B.9 Intuitive Analogy: Counting the Same Box Twice

Imagine moving boxes into storage.

If one box is counted in three rooms at once, the inventory is wrong.

The same problem can happen with channels. A structure may be fragmented, scale-local, and threshold-sensitive. Paper 150N requires a clear accounting rule so the same positive stretching is not counted multiple times.

B.10 Intuitive Analogy: Sparks That Keep Reappearing

Imagine sparks that appear, disappear, and reappear very quickly.

Each spark may seem harmless. But if infinitely many sparks appear, the total danger can grow.

That is threshold flicker or burst behavior. Paper 150N says the constants from these bursts must be summable. Otherwise, small events can accumulate into a big problem.

B.11 Intuitive Analogy: Broken Ice on a River

Imagine a sheet of ice breaking into pieces on a river.

If the pieces separate and melt, the sheet loses strength. But if the pieces jam together and move as one, the danger remains.

Fragmentation in fluid stretching is similar. Breaking apart may help, but only if it creates cost or destroys coordination.

B.12 Thirty-Word Summary

Paper 150N studies broken, filtered, and threshold-shifting stretching: fragmentation, scale-locality, and complement routes must pay cost, become lower-order, or become residual pathology.